Probabilistic Investigation of Sensitivities of Advanced Test-Analysis Model Correlation Methods

Liz Bergman, Matthew S. Allen, and Daniel C. Kammer
Dept. of Engineering Physics
University of Wisconsin-Madison
Randall L. Mayes
Sandia National Laboratories
FEM mass matrix must be reduced to test degrees of freedom (TAM) in order to compute modal orthogonality.
The Controversy

- Current FEM reduction algorithms
  - Static TAM: fails for heavy, soft structures. May be difficult to achieve good TAM/FEM correlation
  - Improved Reduced Static (IRS) TAM: ill-conditioned under certain circumstances
  - Modal TAM: Trivial to achieve perfect TAM/FEM correlation, however it has a reputation of being highly sensitive to experimental or modal-mismatch errors
Purpose of Research

- Study the sensitivity of various TAMs to gain insight into factors that strongly affect sensitivity
- A probabilistic analysis will be used to characterize the effect of measurement errors on TAM sensitivity
Relevant Literature

- Freed, AM and Flanigan, CC (1990): Modal TAM most sensitive, sensors placed using modal kinetic energy
- Avitabile, P, Pechinsky, F, and O’Callahan, J (1992): Sensor placement is vital to TAM performance, SEREP and Hybrid perform better than Static TAM for small sensor sets
- Chung, YT (1998): Sensor placement was not discussed and no significant difference could be seen between the TAMs
Relevant Literature

- Gordis, JH (1992), Bleloch, P and Vold, H (2005):
  - Notes ill-conditioning in dynamic reduction equation:
    \[
    \{ \phi_{io} \} = - \left[ K_{oo} - \omega_i^2 M_{oo} \right]^{-1} \left[ K_{oa} - \omega_i^2 M_{oa} \right] \{ \phi_{ia} \}
    \]
  - Proposes that IRS TAM will be ill conditioned if the natural frequencies of the structure with the o-set DOF pinned are similar to the frequencies of the structure of interest.
  - Recently, this theory seems to have been applied to other TAM techniques such as the Modal TAM.

Figure 1. The NASA/Langley 10-Bay Truss.
Model

- Generic Satellite
  - 7,146 DOF
  - Target modes: first 18 consecutive flexible modes (0.3-11.8 Hz)
  - 108 sensors

Target Mode 5
2.7 Hz

Target Mode 6
2.8 Hz

Target Mode 7
3.5 Hz

Target Mode 8
3.7 Hz
Test Analysis Models – Static TAM

\[ a = \text{sensor location} \]

\[ o = \text{omitted DOF} \]

**Eigenvalue problem**

\[
-\omega_i^2 \begin{bmatrix}
M_{aa} & M_{ao} \\
M_{oa} & M_{oo}
\end{bmatrix}
\begin{bmatrix}
\phi_{ia} \\
\phi_{io}
\end{bmatrix}
+ \begin{bmatrix}
K_{aa} & K_{ao} \\
K_{oa} & K_{oo}
\end{bmatrix}
\begin{bmatrix}
\phi_{ia} \\
\phi_{io}
\end{bmatrix}
= 0
\]

**Lower partition equation**

\[
\begin{bmatrix}
K_{oa} - \omega_i^2 M_{oa}
\end{bmatrix}
\begin{bmatrix}
\phi_{ia}
\end{bmatrix}
+ \begin{bmatrix}
K_{oo} - \omega_i^2 M_{oo}
\end{bmatrix}
\begin{bmatrix}
\phi_{io}
\end{bmatrix}
= 0
\]

**Neglect the mass of the o-set DOF**

\[
\begin{bmatrix}
\phi_{io}
\end{bmatrix}
= -\begin{bmatrix}
K_{oo} - \omega_i^2 M_{oo}
\end{bmatrix}^{-1}
\begin{bmatrix}
K_{oa} - \omega_i^2 M_{oa}
\end{bmatrix}
\begin{bmatrix}
\phi_{ia}
\end{bmatrix}
\]

**Static Transformation Matrix (each column represents a constraint mode)**

\[
[T_S] = \begin{bmatrix}
I \\
-K_{oo}^{-1}K_{oa}
\end{bmatrix}
\]
Test Analysis Models – IRS TAM

\[ \{ \phi_{io} \} = - \left[ K_{oo} - \omega_i^2 M_{oo} \right]^{-1} \left[ K_{oa} - \omega_i^2 M_{oa} \right] \{ \phi_{ia} \} \]

Ill-conditioned when \( \omega_i^2 \) is near any of the eigenvalues of the \( K_{oo}, M_{oo} \) system

Approximate the frequency terms

\[ \omega_i^2 \{ \phi_{ia} \} = \tilde{M}_S \tilde{K}_S^{-1} \{ \phi_{ia} \} \]

Calculate the IRS transformation matrix

\[
[T_{IRS}] = [T_S] + [T_i] \\
[T_i] = -\begin{bmatrix} 0 & 0 \\ 0 & -K_{oo}^{-1} \end{bmatrix} \begin{bmatrix} M_{aa} & M_{ao} \\ M_{oa} & M_{oo} \end{bmatrix} \begin{bmatrix} I \\ -K_{oo}^{-1} K_{oa} \end{bmatrix} \tilde{M}_S^{-1} \tilde{K}_S
\]
Test Analysis Models – IRS TAM

O-set system Mode 1
16.8 Hz

FEM Target Mode 18
11.8 Hz
Test Analysis Models – Static and IRS TAM

- Mass weighted effective independence did not select the lumped masses (the lumped masses were essential to TAM-FEM correlation)
- Modal kinetic energy applied to all 18 target modes was not sufficient
- A significant amount of hand selection and engineering judgment was used (modified modal kinetic energy method)

5 core lumped masses
Test Analysis Models – Modal TAM

Physical coordinates in terms of modal coordinates

\[
\begin{bmatrix}
{x_a} \\
{x_o}
\end{bmatrix} = \begin{bmatrix}
\phi_a \\
\phi_o
\end{bmatrix}\{q\}
\]

Partitioned Equations

\[
\begin{bmatrix}
{x_a} \\
{x_o}
\end{bmatrix} = \begin{bmatrix}
\phi_a \\
\phi_o
\end{bmatrix}\{q\}
\]

Solve for modal coordinates in terms of the sensor DOF

\[
\{q\} = \begin{bmatrix}
\phi_a^T \\
\phi_a
\end{bmatrix}^{-1} \phi_a^T \{x_a\}
\]

Modal transformation matrix

\[
[T_M] = \begin{bmatrix}
I \\
\phi_o (\phi_a \phi_a)^{-1} \phi_a^T
\end{bmatrix}
\]
Sensor placement achieved with Effective Independence

Maximize the determinant of the Fisher information matrix

\[
\max \|Q\| = \max \|\phi_a^T \phi_a\|
\]

Effective Independence

\[
E_{Di} = \phi_i^T Q^{-1} \phi_i \\
0.0 \leq E_{Di} \leq 1.0
\]
Modal TAM o-set frequencies are similar to the FEM frequencies, so the theory of Gordis suggests that this TAM will be sensitive.
Modal coordinates in terms of the sensor DOF
\[
\{q\} = \left[\phi_a^T \phi_a\right]^{-1}\phi_a^T\{x_a\}
\]

Solution is more sensitive if the condition number of \(\phi_a\), is large.

Begin with a visualization set, and add sensors that minimize the condition number of \(\phi_a\)
Select sensor locations and reduce FEM

7,146 DOF FEM

Update FEM

108 DOF TAM

TAM/FEM Correlation

TAM/Test Correlation

Fails

Passes

Fails

Update FEM

TAM/Test Correlation

Fails

Passes

108 Sensor Test

Orbital Sciences

TAM/FEM Correlation

FEM validated

passes

fails
Correlation Metrics

☑ Orthogonality

☐ Criteria:  $0 \leq \text{off diagonal term} \leq 0.1$

\[
O = \left[ \phi_{FEM} \right]^T \left[ \tilde{M}_{TAM} \right] \phi_{FEM}
\]

☑ Cross Orthogonality

☐ Criteria:  $0 \leq \text{off diagonal term} \leq 0.1$

$0.95 \leq \text{diagonal term} \leq 1.0$

\[
CO = \left[ \phi_{FEM} \right]^T \left[ \tilde{M}_{TAM} \right] \phi_{TAM}
\]

☑ Frequency Comparison

☐ Criteria:

\[
f_{\text{error}} = \frac{f_{\text{FEM}} - f_{\text{TAM}}}{f_{\text{FEM}}} \times 100 \leq 3\%
\]
TAM-FEM Correlation

Static TAM Orthogonality Matrix

Max off diagonal term: 0.05

IRS TAM Orthogonality Matrix

Max off diagonal term: 6e-4

*Modal TAM always produces perfect orthogonality for TAM-FEM correlation
Test Analysis Correlation

Select sensor locations and reduce FEM

7,146 DOF FEM

Update FEM fails

TAM/Test Correlation passes FEM validated

108 DOF TAM

108 Sensor Test

TAM/FEM Correlation passes

Orbital Sciences
Noise Model and Simulated Test Mode Shapes

\[
\{\phi\}^i_{FEM} = \begin{bmatrix}
\phi_1^i \\
\phi_2^i \\
\vdots \\
\phi_{n-1}^i \\
\phi_n^i 
\end{bmatrix}
\]

Max value

\[
\{\phi\}^i_{Test} = \{\phi\}^i_{noise} + \{\phi\}^i_{FEM}
\]

\[
\{U\} = \text{column vector of uniformly distributed random numbers between -1 and 1}
\]

\[
*2\%* \{U\} = \{\phi\}_{noise}
\]
Noise Model

- FEM assumed to be perfect
- Noise vector models the net effect of all errors that cause the FEM mode shapes to disagree with the test mode shapes.
  - Noise Distribution: Uniform – no assumption is made about the distribution of noise
  - Noise Amplitude: Sensors with the smallest motion have the largest noise to signal ratio
  - Noise is small on average: ± 2% at sensor locations with the largest motion.
TAM-Test Correlation Results (1 case of Random Noise)

Static TAM Orthogonality Matrix

Max off diagonal term: 0.27

IRS TAM Orthogonality Matrix

Max off diagonal term: 0.79
TAM-Test Correlation Results
(1 case of Random Noise)

Max off diagonal term: 0.05

Max off diagonal term: 0.04
Monte Carlo Simulation

- Thus far, TAM-Test correlation has been studied using only one noise profile

- Random noise added in 10,000 iterations

- Orthogonality computed for each iteration

- Maximum off-diagonal term of orthogonality was stored
Despite its low o-set frequencies, Modal TAM does not show high sensitivity!
TAM-Test Correlation Results

- If Orthogonality > 0.1 one might
  - Refine FEM before exiting test
  - Repeat test and/or look for errors
  - Update the FEM

- In this case, the FEM was perfect (errors in test modes were purely random)

- Note: The specific ranking of different TAM methods may depend on:
  - The structure of interest
  - The characteristics of the noise
  - Systematic errors between the test and FEM
Sensor selection is critical to the performance of each TAM.

Most previous studies used the same sensor set, usually optimized for the Static TAM.
Predicting Standard Deviation

- Recently, we have developed formulas to analytically predict sensitivity of a TAM based on simple metrics.
- For example, for the noise model used in this study:

\[ O_{ij} = \left[ \phi_i + n_i \right]^T \left[ \tilde{M}_{TAM} \right] \left[ \phi_i + n_i \right] \]

\[ n_i = \text{noise} \]

\[ \sigma(O_{ij}) = \sqrt{\sum_m \left( \tilde{M}_{TAM} \phi_j \right)_m^2 \sigma_i^2 + \sum_m \left( \phi_i^T \tilde{M}_{TAM} \right)_m^2 \sigma_j^2 + \sum_m \sum_n \left( \tilde{M}_{TAM} \right)_{mn} \sigma_i \sigma_j} \]

<table>
<thead>
<tr>
<th>Maximum Orthogonality Off-Diagonal</th>
<th>Predicted STD</th>
<th>Actual STD</th>
</tr>
</thead>
<tbody>
<tr>
<td>Static</td>
<td>0.03</td>
<td>0.03</td>
</tr>
<tr>
<td>Modal EFI</td>
<td>0.009</td>
<td>0.01</td>
</tr>
<tr>
<td>Modal C#TAM</td>
<td>0.006</td>
<td>0.006</td>
</tr>
<tr>
<td>Modal with Static DOF</td>
<td>0.02</td>
<td>0.02</td>
</tr>
</tbody>
</table>
Conclusions and Future Work

☐ Conclusions
  ☐ IRS TAM was ill-conditioned, as predicted by Gordis
  ☐ Modal TAM did not show high sensitivity even though its o-set frequencies were near those of the target modes
  ☐ Probabilistic analysis more fully explains TAM sensitivity
    ☐ One can even predict the sensitivity of the TAMs analytically given the TAM Mass matrix, mode shapes and noise model.

☐ Future Work
  ☐ Develop more accurate noise models
  ☐ Study the effect of systematic mismatch between FEM and test due to modeling errors.
    ☐ May need the Hybrid TAM in these cases
  ☐ Apply these methods to other physical systems, analytically and experimentally.
    ☐ Investigate systems with non-consecutive target modes