A NEW EXPERIMENTAL METHOD FOR NONLINEAR SYSTEM IDENTIFICATION
BASED ON LINEAR TIME PERIODIC APPROXIMATIONS

by

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Abstract

This work presents a new nonlinear, experimental system identification technique, dubbed the Nonlinear-Linear Time Periodic (NL-LTP) identification method, that can be applied to systems that are excited with a periodic input force and therefore respond with periodic motion. The system is first forced to oscillate in stable periodic orbit, and then a small disturbance force is used to perturb the system slightly from that orbit. One then measures the response until the system returns to the periodic orbit. If the nonlinearities in the system are sufficiently smooth and the perturbation from the periodic orbit is sufficiently small, then one can approximate the system model as linear time periodic about the periodic orbit. One of a variety of methods can then be used to extract the time varying modal model of the system from the response about the limit cycle. The extracted modes are used to construct a time periodic state transition matrix and state coefficient matrix, which describe the system’s nonlinear dynamics over a range of the states. Finally, a method is derived to construct the underlying nonlinear system model, or nonlinear restoring forces, from the linear time periodic model that was identified experimentally.

The resulting model for the nonlinear system encompasses that portion of the state space that is traversed by the system during its periodic orbit. The method is valid for higher order systems and it is attractive because it allows one to identify the order of the nonlinear system and the nonlinear system model without assuming the mathematical form for the nonlinearities a priori.

This dissertation will also explore the NL-LTP method by applying it to several systems, including a simulated Duffing oscillator and two degree-of-freedom Galerkin model of a nonlinear cantilever beam, as well as an actual continuous cantilever beam with a geometric
nonlinearity that is driven in a periodic limit cycle by an electromagnetic shaker. Since harmonically forced nonlinear systems can bifurcate for certain forcing configurations, a numerical continuation technique was also developed to compute the nonlinear frequency response of the models over large frequency ranges. These response curves, which define the systems' periodic orbits, were used to catalog all the periodic orbits that were possible. Then, these curves were used to choose a variety of forcing configurations to apply the identification and assess the ability of the method to accurately estimate the system model based on the character of the periodic orbit. Based on these results, a set of guidelines for the optimal excitation conditions is provided. Furthermore, this work presents several tools that are needed in order to effectively apply the NL-LTP method for multi-degree-of-freedom systems. This process can be sensitive to spurious terms in the identified linear time periodic model, so metrics are presented that aid the user in determining which terms are spurious and should be eliminated. A new procedure is also presented that allows one to validate the identified model by comparing its restoring forces with the total restoring force estimated using the restoring force surface method of Masri and Caughey.
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Nomenclature and Abbreviations

Nomenclature

\( a, b \) scalar constants
\( f \) nonlinear equation of motion function
\( h \) nonlinear output equation of motion function
\( j, k, l, m \) scalar integers
\( n, p, q, r \) scalar integers
\( t \) time variable
\( u \) input vector
\( x \) state vector
\( y \) output vector

\( L \) matrix of constants
\( M \) mass matrix
\( R \) transformation matrix
\( S \) transverse section
\( T \) matrix transpose operator
\( T \) period
\( U \) open set
\( V \) local neighborhood
\( W \) subset

\( f_{i,j}^x \) integral of \( i,j^{th} \) component of state coefficient matrix
\( \hat{f} \) estimate for nonlinear equation of motion function

\( \gamma \) periodic orbit

\( g_{i,j}^{*} \) \( i,j^{th} \) nodal restoring acceleration

\( g_{RF} \) vector of total restoring accelerations

\( \lambda_r \) \( r^{th} \) (Floquet) eigenvalue

\( \phi_r(t) \) \( r^{th} \) right (Floquet) eigenvalue

\( \psi_r(t) \) \( r^{th} \) left (Floquet) eigenvalue

\( \tau \) constant time variable

\( \bar{u} \) periodic input vector

\( \hat{u} \) perturbation input vector

\( \omega \) circular frequency variable

\( \omega_T \) periodic orbit frequency

\( \bar{x} \) periodic state vector

\( \tilde{x} \) perturbation state vector

\( x_d \) displacement response vector

\( x_v \) velocity response vector

\( x_a \) acceleration response vector

\( y^L \) lifted output vector

\( y_m^L \) \( m^{th} \) sample of the lifted output vector

\( A(t) \) state coefficient matrix
\[ B(t) \] input distribution matrix

\[ \{ B_m \}_r \] \( m^{th} \) Fourier coefficient vector of the \( r^{th} \) mode

\[ \{ \bar{B}_m \}_r \] \( m^{th} \) left Fourier coefficient vector of the \( r^{th} \) mode of an EMP signal

\[ [ B_m ]_r \] \( m^{th} \) Fourier coefficient matrix of the \( r^{th} \) mode

\[ C(t) \] output distribution matrix

\[ \mathbb{C} \] complex number set

\[ C_{i,j} \] \( i,j^{th} \) constant of integration from integrating the state coefficient matrix

\[ \{ \bar{C}_m \}_r \] \( m^{th} \) right Fourier coefficient vector of the \( r^{th} \) mode of an EMP signal

\[ D(t) \] direction throughput matrix

\[ F_{ext} \] external force vector

\[ \hat{G}(\omega) \] harmonic transfer function

\[ G_{RF} \] system restoring force vector

\[ \Lambda \] diagonal matrix of \( \) (Floquet) eigenvalues

\[ N_B, N_h \] truncation integer scalars

\[ P(x,u) \] Poincaré map

\[ \mathcal{P}(t) \] periodic matrix

\[ \Phi(t,t_0) \] state transition matrix

\[ \Psi(t) \] matrix of eigenvectors

\[ \mathbb{R} \] real number set

\[ [ R(t) ]_r \] residue matrix of the \( r^{th} \) mode
\( \{ R_y(t) \}_r \) output residue vector of the \( r^{th} \) mode

\( \{ R^L \}_r \) lifted residue vector of the \( r^{th} \) mode

\( X(t) \) fundamental solution matrix

\( X_f(x,u) \) flow of the differential equation of motion

**Abbreviations**

AMI Algorithm of Mode Isolation

CMC Computed Muscle Control

DO Duffing oscillator

DOF degree-of-freedom

EMP exponentially modulated periodic

FFT fast Fourier transform

FRF frequency response function

FSE Fourier series expansion

HTF harmonic transfer function

LTI linear time invariant

LTP linear time periodic

LTV linear time varying

mod modulo function

MS Orbit multiple solutions orbit

NL nonlinear

NL-LTP nonlinear-linear time periodic
<table>
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<th>Abbreviation</th>
<th>Description</th>
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<tr>
<td>NRCM</td>
<td>Newton-Raphson correction method</td>
</tr>
<tr>
<td>OMA</td>
<td>output-only modal analysis</td>
</tr>
<tr>
<td>SSI</td>
<td>Stochastic Subspace Identification</td>
</tr>
<tr>
<td>STM</td>
<td>state transition matrix</td>
</tr>
<tr>
<td>TAM</td>
<td>test analysis model</td>
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Introduction

Most dynamical systems behave nonlinearly in the most general scenario. This is true of the physical sciences (i.e. physics, biology, and engineering) as well as the social sciences (i.e. economics [1] and psychology [2]). Nonlinear systems are difficult to model from first principles, so one would hope to characterize the system experimentally instead. The basic process is to design an experiment that correctly represents the nonlinear system or nonlinear phenomena of interest. Measurements are then taken at selected coordinates on the experimental structure, and an identification method is applied to the experimental measurements to extract parameters that define physical properties of the experimental structure.

Although the methods used for identifying models of linear systems are quite mature, in many cases linear theory cannot correctly describe some of the phenomena that are exhibited by nonlinear systems. For example, torque free satellite orbits can experience unpredictable and uncontrollable motions (chaos), a cataclysmic nonlinear phenomenon, during in-space attitude maneuvers [3]. Voltage variability in power systems can lead to systemic blackout due to abrupt changes in the nonlinear system dynamics (bifurcations) [4]. The flutter of airfoils due to aeroelastic effects [5] can lead to damaging oscillations in airplane wings. Additionally, material nonlinearities can cause hardening of jet engine-to-pylon connections and lead to catastrophic failure [6]. These nonlinear phenomena cannot be predicted with linear theory or by linearly derived models. Nonlinear models must be used to correctly predict the behavior of such systems, and this requires the use of nonlinear system identification techniques. The rest of this section will introduce the important concepts for this dissertation. The current methods for nonlinear system identification will be reviewed, and then the new nonlinear identification method will be explained. Then, since the new method requires periodic motion, the conditions
for periodic motion and the stability and character of periodic motion will be discussed. Finally, the scope and format of the dissertation will be discussed.

1.1 Review of Existing Nonlinear System Identification Techniques

Various nonlinear system identification methods have emerged to attempt to extract mathematical models of nonlinear systems from measurements [6]. However, with current methods it is difficult to extract meaningful information from measurements if the nonlinear system’s order is high or if the form of the nonlinearities is not known a priori.

For example, the restoring force surface (RFS) method by Masri and Caughey [7] extracts a time domain model of the system based on its internal restoring force. The method uses the system's measured acceleration to estimate the velocity, displacement, and restoring forces of the system. The restoring forces are plotted as a surface against the displacement and velocity, and a restoring force model is fit to this surface. Although the method has been valuable, for example in the automotive industry [8-10], it can be difficult to apply to higher order systems.

Another widely used pair of methods are the Hilbert transform and Hilbert-Huang transform (HHT). These approaches can be used to detect the instantaneous frequency and amplitude of a time domain signal. If a signal has multiple frequency components then it is first decomposed into single frequency component signals with the Hilbert-Huang method [11], which is also called empirical mode decomposition, and then the Hilbert transform can be used to estimate the time varying nonlinear parameters. Several algorithms have been suggested for these methods over the years [12-14], so they are fairly well known. However, the Hilbert-Huang method is not always successful, for example if the signal contains frequencies that differ
by less than factor of two [14], and a extensive amount of signal processing is required in order to apply these methods to multi-degree-of-freedom systems.

Another approach that has been studied is the nonlinear auto-regressive moving average with exogeneous input (NARMAX) method, which involves fitting polynomial coefficients to a discrete time series representation of the response of a system. Very accurate fits can be achieved with the method, especially since it incorporates a moving average that reduces the effects of extraneous noise in the signal. A major disadvantage of the technique is that a very high order fit may be required if the nonlinearities of the system are not of a polynomial form. Alternatively, one could use nonpolynomial fitting terms, but that requires knowing or assuming the form of the nonlinearity, which one would like to avoid. Furthermore, by fitting the coefficients of a discrete-time model, the model that this technique extracts provides little insight to physical interpretation of the parameters [6].

Some researchers have applied the nonlinear subspace identification [15], which is available for multi-degree-of-freedom systems in the time domain. Using this method, one assumes the form of the nonlinearities as a function as the input applied to the system and the measurements of the system. This method has similar drawbacks to the previous method.

A number of frequency domain techniques are also available such as the Volterra and Weiner series. The Volterra and Weiner series is attractive because it can be used to determine a higher order relationship between the input and output signals for a nonlinear system. Essentially, this is a higher order version of the linear frequency response function (FRF). A drawback of this approach is that one requires extremely large amounts of data to get good estimates for even 2nd or 3rd order nonlinearities. Furthermore, this method is nonparametric and it is not valid near abrupt changes in the dynamics of a system.
Two other frequency domain approaches are the *conditioned reverse path* (CRP) method and *nonlinear identification through feedback of the output* (NIFO) method [6]. The CRP method is attractive because it is suitable for multiple DOF systems and allows one to extend the linear frequency response function concept to nonlinear systems, but it requires one to assume a form of the nonlinearities that one is trying to identify. Basically, one assumes the form and location of the nonlinearities and treats them as inputs to the system. Then, one estimates frequency response functions with respect to all the inputs in order to identify the underlying linear frequency response functions and the nonlinear parameters. Other drawbacks include that the nonlinearities must be isolated and one is required to measure the structure at the nonlinearities. The NIFO method is very similar to the CRP method, but is simpler theoretically. It was developed in parallel and has similar advantages and disadvantages to the CRP method. Additionally, some studies suggest the NIFO method is not as robust as the CRP method to ill-condition matrices in the calculation of the model [6].

Many researchers have recently attempted to calculate systems' *nonlinear normal modes* (NNMs) Rosenberg, 1960 #83; Shaw, 1993 #93; Boivin, 1995 #95; Vakakis, 1992 #128}. There are several definitions of NNMs, but one that is widely used defines an NNM as a non-necessarily synchronous periodic solution of the unforced and undamped nonlinear system (i.e. essentially a resonance condition) [16]. The experimental work with NNMs exploits the fact that the resonant conditions of a nonlinear system evolve with energy. Therefore, one can observe NNMs in experiment by exciting a system to a high state of energy and then removes the excitation to allow the system to dissipate energy until it returns to a low energy state. A processing technique called the *wavelet* transform, which tracks the frequency of the response versus the energy in the system, can be applied to the signal, essentially providing the backbone
curve [17] of the NNM. The method has shown to provide results that are comparable to analytical methods, but the experimentalist has no control over the decay response, and the system response may jump between different NNMs [18], making the results more difficult to interpret. Furthermore, the method is better suited to verify that a computed NNM model is accurate rather than to identify a model for a system from measurements.

More information on many of these methods can be found in a few recent review papers [6, 19, 20]. While these methods are successful in some situations, when the order of the system is higher than two or when the form of the nonlinearities is not known a priori there are few methods that are effective. In this work, a new method is considered that overcomes some of these limitations. The method does not require one to guess the mathematical form of the system’s nonlinearities a priori and it is potentially applicable to relatively high order systems. The method is based on approximating the system as linear time periodic about a limit cycle so that simpler identification methods can be exploited.

1.2 Proposed Identification Method

Figure 1 outlines the proposed NL-LTP approach. The system of interest is assumed to be oscillating in a stable periodic orbit. (In an experiment such a limit cycle might be obtained by driving the system periodically with an electromagnetic shaker.) A small perturbation is then applied causing response to deviate somewhat from its periodic orbit, but since the orbit is stable the response of the system eventually returns to the periodic orbit once again. The response of the system is recorded as it returns to the orbit and this perturbation is the primary data used to perform system identification. Since the orbit is periodic, the perturbations from the periodic orbit can be modeled using a linear time periodic system model. Thus, experimental
identification techniques for linear time periodic systems are used to extract the time periodic modes of the system from the measurements. In this work, the lifting method [21] is used together with the frequency domain Algorithm of Mode Isolation (AMI) [22] for this purpose. For some systems the Stochastic Subspace system identification (SSI) method [23] or other methods may be preferred instead of AMI, and an example of using the SSI method is also explored and found to give similar results. Under slightly different forcing conditions, a variant of the Harmonic Transfer Function (HTF) [24] that was proposed by Allen et al. [25] can also be used to identify the time periodic system, and this is also discussed. Once the time periodic modes have been estimated, the state transition matrix and state coefficient matrix of the linear time periodic system can be reconstructed, as described in a recent work by Allen [21]. Because the system was linearized about the entire nonlinear periodic orbit, the resulting state matrix describes the nonlinearities at each point in the state space that was visited during the periodic orbit. This model can then be integrated to directly estimate the nonlinear functions that comprise the equations of motion over that region of the state space. These nonlinear functions describe, for example, the nonlinear force-displacement relationships between the different nodes in the system. This flow of this procedure is provided in Fig. 1.

![Figure 1: Overview of the nonlinear system identification technique [26].](image-url)
This procedure has the following important advantages over other existing identification routines:

1. The size of the estimated state coefficient matrix provides a direct estimate for the order of the system, and the time periodic terms of the state coefficient matrix contain the functions of the nonlinear parameters.

2. The model is identified for each instant in the entire cycle of the periodic response. Therefore, the nonlinear parameters are also estimated for each instant in the entire periodic response, and the function form of the nonlinearities can be easily constructed over the range of state space visited by the system during the periodic response.

The second point is a stark improvement over methods that directly linearize the nonlinear response and perform linear time invariant identification. In that case, the model is estimated only at a single point in the state space. Here, the model has been effectively linearized over an entire trajectory, and then the identification provides a model that is valid for every state within the bounds of that trajectory.

1.3 Nonlinear Systems and Periodic motion

In nonlinear dynamics, periodic responses tend to be an inherent system property for many nonlinear systems. For example, inverted pendulum models have been used to model the passive dynamics of human gait (i.e. walking and running) [27]. These models have nonlinear equations of motion due to large angle rotations of the pendulum legs and due to the pendulum leg impacting the ground. When the models are initiated to walk down a sloped surface with the appropriate conditions, they tend to walk in a periodic orbit. In this case, the periodic motion is inherently linked to the nonlinear dynamics of the system.
Nonlinear systems that are not inherently periodic can be driven with a periodic input force so that they respond in a periodic orbit. Rotor dynamics systems are a common example. Many researchers have studied nonlinearities in rotor systems due to interaction between the rotor and its clearance bearing [28]. Wind turbines are another application of rotor dynamics, and complex nonlinear instabilities can occur at certain forcing frequencies and amplitudes arising from the time-periodic effects of the system [29].

Even if a system does not exhibit natural periodic motions or operate under periodic forcing, it is typical to perform experimental vibration testing using harmonic forcing (i.e. sine sweep testing). In summary, many if not most nonlinear systems, respond naturally or can be driven to respond periodically. When this is the case, the proposed methods of this work can be applied, so long as the periodic orbit is stable.

1.4 Stability of Periodic Orbits

This method is only valid for system that have at least one stable periodic orbit. As mentioned previously, a stable orbit can be observed in experiment if the periodic response is perturbed, and the system recovers to the original periodic response. Theoretically, there are many techniques for checking stability, and most techniques for time periodic methods involve the use of Floquet theory [30] to calculate the eigenvalues of the Monodromy matrix [31]. The Floquet theory will be summarized in Section 2.4.2, but informally, the most general idea of stability is the following: a periodic orbit is stable when a small applied perturbation does not eject the response to a different dynamic response configuration. In this work, a slightly stricter version of stability is required. Here, the systems are required to be asymptotically stable, which means that after a small perturbation is applied to a periodic orbit, the resulting response will
tend to return to the periodic orbit with increasing time. A periodic orbit is unstable if an arbitrarily small perturbation ejects the response to a completely different dynamic response configuration. The proposed identification method is not applicable in this case because the system is likely to exhibit large motions as it jumps from one periodic orbit to the next so that the resulting response cannot be well described by a model that is linear about the periodic orbit.

1.5 Periodic Solutions in a Variable Parameter Spaces

For systems that exhibit periodic motion, the stability property and other properties such as magnitude and complexity depend on the parameter values that describe the system. For example, in systems that are excited with periodic forcing, the periodic response will change drastically as the forcing frequency and amplitude is changed. Furthermore, periodic orbits can disappear, additional periodic orbits can appear, and the stability of the orbits can change when the forcing frequency is changed. So it is important to understand what periodic orbits are possible and how the periodic orbit properties affect the identification.

In order to understand these ideas for a given system, a numerical continuation technique will be used to calculate the systems' periodic orbits for different forcing configurations. The method iteratively calculates a branch of periodic solutions as the forcing frequency is varied. Several mature continuation codes exist, such as AUTO [32] and MATCONT. However, these algorithms are cumbersome to use for forced response. Kerschen, Peeters, and their associates are developing an efficient continuation algorithm that is well suited for problems in structural dynamics [33]. However, their code is constrained to calculate the nonlinear normal modes (unforced and undamped periodic solutions) and has no mode of operation for forced response. This work uses a slightly different approach for calculating the periodic solutions of the forced
responses of the systems. This will be presented later. More importantly, the periodic solution branches, which are essentially frequency responses, will be used to determine what conditions are needed to achieve a successful NL-LTP identification.

1.6 Scope of the Dissertation

The rest of this dissertation is organized as follows. Section 2 provides the necessary background on systems theory including nonlinear systems and linear systems. Section 3 introduces the linear time periodic modeling theory that is needed to perform the extraction and the nonlinear modeling theory that is need to estimate the nonlinear system. In Section 4, the prescribed nonlinear system identification method is performed on the simulated measurements of a Duffing oscillator and a discussion of the results is presented. Then, Section 5 addresses how to select optimal forcing configurations with which to apply the identification. This includes a discussion of system bifurcations in Section 5.2. Section 5.3 presents a tool that can be used to calculate the nonlinear frequency responses of a system, and then the frequency responses (and bifurcations) of the Duffing oscillator are presented. Lastly in this section, 5.4 will explore the NL-LTP method for various periodic orbits of the Duffing oscillator at key regions in the nonlinear frequency responses. Then, in Section 6 the entire NL-LTP methodology will be applied to simulated measurements from multi-degree-of-freedom nonlinear beam and actual experimental measurements from a cantilever beam with a nonlinear tip spring. Conclusions will be provided in Section 7 and some recommendations for future work will be discussed in Section 8.
2 Systems Theory

2.1 Introduction

The system identification scheme that is proposed in this work approximates the nonlinear system as linear time periodic (LTP) and so it is necessary to understand nonlinear and linear systems theory. The development will start with general systems. For this work, it is assumed that many systems of interest operate in the presence of some input, and the theoretical development will be based on this assumption. However, most of the theory is applicable to unforced systems by simply setting the input variables to zero. After defining a general nonlinear system using this assumption, it will be shown how the system can be linearized to a linear time invariant (LTI) or linear time periodic approximation. Then, the linear theory that is needed to describe the linearized systems is provided.

2.2 Systems as Functions, Vector Fields, and Maps

The dynamics of a system may be described mathematically as

\[ \dot{x} = f(x,t,u) \, . \] (1)

In this equation, \( f \) is a smooth and differentiable function that describes how the state of the system \( x \), the inputs applied to the system \( u \), and time \( t \) combine to influence the dynamics of the system. Then, the function \( f \) is equal to the time rate of change of the state, \( \dot{x} \). Let \( U \) be an open set such that \( U \subseteq \mathbb{R}^n \) (i.e. \( U \) is contained in \( \mathbb{R}^n \)). Then, \( x = x(t) \in \mathbb{R}^n \) (i.e. \( x \) is a member of \( \mathbb{R}^n \)) and \( u = u(t) \in \mathbb{R}^p \) are vector valued functions of time, and let the function \( f : U \rightarrow \mathbb{R}^n \) (i.e. \( f \) maps from the set \( U \) to \( \mathbb{R}^n \)) be a smooth function [34]. Then, the function \( f \) is said to generate
a flow. Let \( X_t : U \rightarrow \mathbb{R}^n \) be the flow for the function \( f \). The flow \( X_t(x,u) = X(x,t,u) \) is a function that depends on both the state vector and the input, and it is assumed to be \( C^1 \), or at least one time continuously differentiable on its interval \([35]\). If it also satisfies Eq. (1),

\[
\frac{d}{dt}(X_t(x,u)) \bigg|_{t=\tau} = f(X_\tau(x,u))
\]

then it can be used to describe all the solution curves of the Eq. (1). For example, for initial conditions \( x_0 \) and \( u_0 \) contained in \( U \), the solution of Eq. (1) at time \( t \) contained in some interval \( I = (t_1, t_2) \subseteq \mathbb{R} \) is defined as \( X_t(x_0,u_0) \).

For certain practical applications, these mathematical assumptions can be carefully relaxed whilst still providing a good theoretical basis for these systems as well. For example, dynamics of certain systems may contain jump-discontinuities or kinks in vector field that describes the differential equation. If this is the case, then the flow is assumed to be locally \( C^1 \), even though it may not be globally \( C^1 \) in the full subset of interest.

Often in practice, one measures the output of a system \( y = y(t) \in \mathbb{R}^q \), which is usually some set of the states of the system such that \( q \) is not necessarily equal to \( n \).

\[
y = h(x,t,u)
\]

The function \( h: U \rightarrow \mathbb{R}^q \) is also assumed to be a smooth and differentiable function. Together Eqs. (1) and (3) fully define a given system and its dynamics. If continuous time is sampled impulsively at discrete, even intervals \( \Delta t \) and the \( k^{th} \) sample occurs at \( t_0 + (k-1)\Delta t \), then the discrete counterparts to Eqs. (1) and (3) are provided by the following maps.
The orbits defined by the maps in Eq. (4) are the sequences \( \{x_k\} \) and \( \{y_k\} \) where \( k \) maps over \( \mathbb{Z} \).

### 2.2.1 The Unforced System

The development of the system differential equations is also valid for the unforced system case and can be derived by setting the input to zero, \( u = \{0\} \). Then, using all the same assumptions for the domain of interest, the vector function, \( x(t) \), and the function \( f \), the following equations describe the continuous time unforced system.

\[
\dot{x} = f(x) \\
y = h(x)
\] (5)

Moreover, using the same impulsive sampling as with Eq. (4), the unforced system maps are the following equations.

\[
x_{k+1} = f(x_k) \\
y_{k+1} = h(x_k)
\] (6)

### 2.3 Nonlinear Systems

A nonlinear system is best understood by first defining the properties of linear systems. Generally, a linear system must obey the laws of *additivity*, *homogeneity*, and *superposition*, which will all be fully defined in the next section. By contrast, a nonlinear dynamical system is any system that is not linear.
Nonlinear systems are qualified and understood by first studying their equilibrium points (or fixed points for maps). An equilibrium of a nonlinear continuous time system is any state $\overline{x} \in U$ and associated input $\overline{u} \in U$ that is constant with time and so satisfies $f(\overline{x}) = 0$ [34, 35]. For a map, the fixed point is any point that the discrete system maps to itself as $f(\overline{x}_k) = \overline{x}_k$. (For the following discussion, the definitions and explanations for equilibria of continuous systems can also be applied in an analogous form to fixed points of discrete systems.)

Stability is one very important property of equilibria. In particular, suppose in a small region $W \subset U$ such that $\overline{x} \in W$ that all solutions $x(t)$ starting in $W$ (i.e., $x(t_0) \in W$) stay in $W$ for all $t > 0$, then the equilibrium is considered stable. Furthermore, if those solutions tend towards the equilibrium $\overline{x}$ as time tends to positive infinity, then the equilibrium is asymptotically stable. If at least one solution starting in the neighborhood of the equilibrium does not remain in $W$ for all time, then the equilibrium is unstable [34, 35]. More generally, if the equilibrium is not stable then it is unstable. Hirsch and Smale [35] present a thorough discussion and proofs of the stability of dynamical system equilibria.

The most typical initial approach for studying nonlinear systems is to study dynamics in small regions near equilibria. Let $(\overline{x}, \overline{u})$ be an equilibrium of the system, where $\overline{x}$ is a constant state vector and $\overline{u}$ is a constant input. Let $(\tilde{x}, \tilde{u})$ be small deviations of the response from the equilibrium conditions (i.e., $\tilde{x} = x - \overline{x}$ and $\tilde{u} = u - \overline{u}$). Then, Eq. (1) can be expanded in a Taylor series, which can be truncated to only the linear terms resulting in the following.

$$\dot{\tilde{x}} = \left[ \frac{\partial f_i}{\partial x_j} \right]_{(\overline{x}, \overline{u})} \tilde{x} + \left[ \frac{\partial f_i}{\partial u_j} \right]_{(\overline{x}, \overline{u})} \tilde{u} \quad (7)$$
The matrices \( \left[ \frac{\partial f_i}{\partial x_j} \right] \) and \( \left[ \frac{\partial f_i}{\partial u_j} \right] \) are the Jacobians of the first partial derivatives of \( f \) that are evaluated at the equilibrium conditions \((\bar{x}, \bar{u})\). The shorthand notation \( \left[ \frac{\partial f_i}{\partial x_j} \right] \) is used for the Jacobian matrix meaning that the component in the \( i^{th} \)-row and \( j^{th} \)-column is the partial derivative of the \( i^{th} \) component of \( f \) with respect to the \( j^{th} \) component of \( x \). The Jacobian matrices in Eq. (7) are constant, and these are linear time invariant models of the linearized system dynamics near \( \bar{x} \), which results from \( \bar{u} \). Furthermore, the linearization is important because the local dynamics, solutions, and stability of solutions of Eq. (7) are usually representative of the original nonlinear system's local properties.

The unforced system may also contain an equilibrium, \( \bar{x} \). When this is the case, the unforced system may also be expanded in a Taylor series, where only the first partial derivative Jacobian matrix is retained. The resulting linearized system has the following form.

\[
\dot{\bar{x}} = \left[ \frac{\partial f_i}{\partial x_j} \right]_{\bar{x}} \bar{x} \tag{8}
\]

### 2.3.1 Nonlinear Systems with Periodic Orbits

Another form of steady state solution is a periodic orbit, and its stability properties strongly influence the flow of the system nearby the periodic orbit. Although systems may respond periodically without the presence of an applied force, it is assumed now that the system is forced with a periodic input, \( \bar{u} \), and therefore responds periodically. A method developed by Poincaré can be used to determine the stability of periodic orbit. Let \( \gamma \) be a periodic orbit of the system with period \( T \) such that the flow \( X_T(\bar{x}, \bar{u}) = \bar{x} \) for each \( \bar{x} \) in \( \gamma \) and under the influence of each \( \bar{u} \). Let \( S \) be a \((n-1)\)-dimensional, local, differentiable, and transverse section (not necessarily planar) of the flow going through \( \bar{x} \) (no trajectory in the flow at \( S \) is tangent to \( S \)),...
and let $V$ be an open, connected neighborhood of $\bar{x}$. Finally, let $\tau$ be the time it takes the flow which originates from $x$ in $V$ to return to $S$. Then, the Poincaré map or first return map $P : V \to S$ is defined by the following equation.

$$P(x,u) = X_{\tau}(x,u)$$ (9)

In particular for the periodic orbit, $P(\bar{x},\bar{u}) = \bar{x}$ and $\tau = T$. The Poincaré map is depicted in Fig. 2.

*Figure 2: Poincaré map for a system.*

$P(x,u)$ is a discrete dynamical system representation of $f$ near the periodic orbit. The point $\bar{x}$ of the periodic orbit is a fixed point on the Poincaré map since $P(\bar{x},\bar{u}) = \bar{x}$. The linearized Poincaré map is formed by calculating the Jacobian matrix $\left[ \frac{\partial P_j}{\partial x_i} \right]$ and evaluating at $(\bar{x},\bar{u})$. Analogous to the general linearization, the stability of the fixed point $\bar{x}$, determined from the eigenvalues of the linearized Poincaré map $\left[ \frac{\partial P_j}{\partial x_i} \right]$, also defines the stability of the periodic orbit $\gamma$. (In Section 2.4.2.1, Poincaré theory will be related to Floquet theory, which is the predominant method for linear time periodic analysis. Both of these approaches can be used to calculate periodic orbit stability.) If the eigenvalues of $\left[ \frac{\partial P_j}{\partial x_i} \right](\bar{x},\bar{u})$ are less than 1 in absolute value, then the fixed point $\bar{x}$ is asymptotically stable [34, 35]. Furthermore, if Poincaré maps
are constructed for hypersurfaces \( S_0, S_1, S_2, \ldots \) all at different transverse crossings of the periodic orbit, then the linearization of all of those Poincaré maps represents a discrete linearization of the nonlinear flow \( X_t(x,u) \) about the periodic orbit \( \gamma \).

It can be difficult to construct the analytical Poincaré map for a particular section of the periodic orbit (or at many sections), so an alternative approach to study periodic orbits and their stability is to use a continuous time linear approximation with the Taylor expansion. Let \( \bar{x} \) be one state contained in the periodic orbit \( \gamma \), and let \( \bar{x} \) be perturbed by a small amount \( \bar{x} \). For example, the perturbation may be introduced by a small disturbance force \( \bar{u} \). Then, the state and the input can be described as \( x = \bar{x} + \bar{x} \) and \( u = \bar{u} + \bar{u} \). Since \( f \) is a \( C^1 \) function, Eq. (1) can be expanded in a Taylor series for these conditions at each instant (i.e. at perturbation for each \( \bar{x} \)) in the periodic orbit. The expansion can be truncated to only the linear terms, which results in the result is the following equation.

\[
\dot{x} = \left[ \frac{\partial f}{\partial x} \right]_{\gamma} \bar{x} + \left[ \frac{\partial f}{\partial u} \right]_{\gamma} \bar{u}
\]

The Jacobian matrices in this equation depend on the state vector and since the state vector within the periodic orbit depends periodically on time, the matrices in Eq. (10) vary periodically with time. Therefore, the system of Eq. (10) is a linear time periodic approximation of the nonlinear system. The main difference then between Eq. (7) and Eq. (10), is that in the latter, the linearization is performed for all the states in the periodic orbit whereas Eq. (7) is only defined for one equilibrium state. Whether for a single state or for all the states within a periodic orbit, once the Taylor approximation has been used to linearize the nonlinear system, the local properties of the nonlinear system can be analyzed using linear techniques.
2.4 Linear Systems

As mentioned previously, a system in the form of Eq. (1) or that has been produced by a linearization of the system in Eq. (1) must obey the properties of additivity, homogeneity, and superposition in order to be defined as linear. Let \( y_1(t) \) be the output of the system from initial conditions \( x_1(t_0) \) and input \( u_1(t) \), and let \( y_2(t) \) be the output of the system from initial conditions \( x_2(t_0) \) and input \( u_2(t) \). Now let \( a, b \in \mathbb{R} \) be constants used to form a new initial condition \( x(t_0) = ax_1(t_0) + bx_2(t_0) \) and a new input \( u(t) = au_1(t) + bu_2(t) \). Then the system is linear if it obeys the properties of additivity, homogeneity, and superposition such that the response is \( y(t) = ay_1(t) + by_2(t) \). The linearization procedures in the previous section produce a system that obeys these properties since \( \frac{\partial f_i}{\partial x_j} \) is evaluated at \( \bar{x} \) and \( \frac{\partial f_i}{\partial u_j} \) is evaluated at \( \bar{u} \), and therefore both of these matrices are constant. One could easily use the initial condition \( x_1(t_0) \) and input \( u_1(t) \) as well as \( x_2(t_0) \) and \( u_2(t) \) to verify the linearity properties with these systems.

2.4.1 Linear Time Varying

The most general form of linear, lumped, state space differential equation of motion is the following equation.

\[
\dot{x} = A(t)x + B(t)u
\]  \hspace{1cm} (11)

In this equation, \( A \in \mathbb{R}^{nxn} \) and \( B \in \mathbb{R}^{nxp} \) are coefficient matrices with potentially time varying coefficients. Each initial state and input pair for Eq. (11) will have a unique solution if the coefficients vary smoothly with time and if the input is integrable [37]. The solution can be written in terms of the state transition matrix. \( \Phi(t,t_0) \in \mathbb{R}^{nxn} \).
The state transition matrix (STM) transfers the state of the system from its initial value at time $t_0$ to its value at a time $t$, and therefore has the following properties.

1. $\Phi(t,t) = I$
2. $\Phi(t_0,t)\Phi(t,t) = \Phi(t,t_0)$
3. $\Phi(t_0,t)\Phi(t_0,t)\Phi(t,t) = \Phi(t,t_0)$

The STM is also a unique solution of $\dot{x} = Ax$ such that

$$\frac{d}{dt} \Phi(t_0,t) = A(t)\Phi(t_0,t)$$

which leads to an efficient computation for constructing the STM. First, one calculates $n$ independent solutions $x_i(t)$ for $i=1,\ldots,n$ to the homogeneous portion, $\dot{x} = Ax$, of Eq. (11) and forms the fundamental matrix $X \in \mathbb{R}^{n \times n}$ as $[X(t)] = [x_1\ x_2\ \cdots\ x_n]$. Then, STM can be computed as

$$\Phi(t_0,t) = [X(t)][X(t_0)]^{-1}.$$  

One must choose the $x_i(t_0)$ to be independent which ensures that $[X(t_0)]$ is nonsingular and therefore invertible.

The measured response of the system may be due to a combined measurement of the state solution (i.e. the homogeneous solution) and some direct influence of the input on the system (i.e. the particular solution) [35, 37, 38]. The measured output is therefore equal to the following equation. The first term on the right hand side contains the homogeneous solution while the second two terms contain the contributions due to the input $u(t)$.
\[ y(t) = C(t)\Phi(t, t_0)x(t_0) + C(t)\int_{t_0}^{t} \Phi(t, \tau)B(\tau)u(\tau)d\tau + D(t)u(t) \quad (16) \]

In the equation, \( C \in \mathbb{R}^{q \times n} \) is an output distribution matrix and \( D \in \mathbb{R}^{q \times p} \) is a direct throughput matrix and both have coefficients that potentially vary with time.

### 2.4.2 Linear Time Periodic

Suppose that all the coefficient matrices in Eq. (11) are periodic with period \( T \) (i.e. \( A(t+T) = A(t) \) or \( B(t+T) = B(t) \)). Then, the system of interest is linear time periodic. This is the continuous time form of the system that results from linearizing a nonlinear system about a periodic orbit as in Eq. (10).

The dynamics of the linear time periodic system can be analyzed using the *Theory of Floquet* [30, 34, 37, 38]. In particular, there is a fundamental matrix \([X(t)]\) for the system, and because the system is periodic, \([X(t+T)]\) is also a fundamental matrix. Using the previous fact, the fundamental matrix can be factored as

\[ [X(t)] = [\overline{P}(t)]^{-1}e^{LT} \quad (17) \]

for a periodic matrix \([\overline{P}(t)]^{-1} \in \mathbb{C}^{n \times n} (\mathbb{R}^{n \times n})\) and a matrix of constants \( L \in \mathbb{C}^{n \times n} (\mathbb{R}^{n \times n})\) [21]. This fundamental matrix can be used to form the STM by Eq. (15) [37, 39].

\[ \Phi(t, t_0) = [\overline{P}(t)]^{-1}e^{L(t-t_0)}[\overline{P}(t_0)] \quad (18) \]

The *Floquet* (or *characteristic*) multipliers are defined as the eigenvalues of \( e^{LT} \), which is the state transition matrix evaluated at \( t = T \) and \( t_0 = 0 \), \( \Phi(T, 0) = e^{LT} \) [34, 38]. The Floquet multipliers of an LTP system are unique [31]. The eigenvalues of the constant matrix \( L \) are called
the Floquet (or characteristic) exponents [34, 38] and are unique only modulo\((i2\pi/T)\) [38]. The matrix \(\Phi(T,0) = e^{LT}\) is also called the Monodromy matrix [40]. Floquet showed that the Floquet multipliers and exponents can be used to determine the stability of the periodic orbit [30]. In particular, if the Floquet multipliers lie inside the unit circle or equivalently if the Floquet exponents have negative real parts, then the periodic orbit is stable.

If all the eigenvalues of \(L\) are simple, then a transformation exists that diagonalizes \(L\) to a matrix of the Floquet exponents. Suppose that \(\Lambda \in \mathbb{C}^{n \times n}(\mathbb{R}^{n \times n})\) is a diagonal matrix of Floquet exponents and \(R \in \mathbb{C}^{n \times n}(\mathbb{R}^{n \times n})\) is an invertible transformation matrix such that \(\Lambda = R^{-1}LR\), then the STM can be formed as

\[
\Phi(t,t_0) = \left[\vec{P}(t)\right]^{-1} e^{RAR^{-1}(t-t_0)} \left[\vec{P}(t_0)\right].
\]

(19)

Since \(\Lambda\) is diagonal, the previous equation can be factored as

\[
\Phi(t,t_0) = \Psi(t)e^{\Lambda(t-t_0)}\left[\Psi(t)\right]^{-1}
\]

(20)

where \(\Psi(t) \in \mathbb{C}^{n \times n}(\mathbb{R}^{n \times n})\) is defined by \(\Psi(t) = \left[\vec{P}(t)\right]^{-1} R\) and is a matrix of time varying eigenvectors for the LTP system [21, 41].

2.4.2.1 Relationship between Floquet Analysis and Poincarè Analysis

Floquet theory is reserved for linear systems but the stability properties from Floquet theory can be related to Poincarè analysis, which is available for both linear and nonlinear systems with periodic orbits. First, let \(\gamma\) be a periodic orbit of the linear time periodic system of interest with \(x \in \mathbb{R}^n\) and with the form of Eq. (11). Let \(P(x)\) be the Poincarè map associated with \(x \in \gamma\) and with the properties described in Section 2.3.1. This Poincarè map is a linear map.
Then, the eigenvalues of the Poincarè map evaluated at \( \mathcal{R} \) are the Floquet multipliers (except for the multiplier of 1) of the Monodromy matrix of the linear time periodic system. Now, suppose that the system of interest is nonlinear and has the form of Eq. (1). Let \( \gamma \) be a periodic orbit of the nonlinear system, and let \( P(x) \) be the nonlinear Poincarè map of a section \( S \) through \( \mathcal{R} \in \gamma \).

Then, the Poincarè map can be linearized about \( \mathcal{R} \), which results in the linear map \( \left[ \frac{\partial P(x)}{\partial x} \right]_{\mathcal{R}} \). The eigenvalues of the linearized Poincarè map are the same eigenvalues that are obtained from the Monodromy matrix, which were defined above as the Floquet multipliers. So, the stability of the periodic orbit can be calculated by: 1) constructing a nonlinear Poincarè map, linearizing the Poincarè map, and calculating the eigenvalues of the linearized map; or 2) linearizing the continuous time nonlinear system to form Eq. (10), solving to find the Monodromy matrix, and calculating the Floquet multipliers of the Monodromy matrix.

### 2.4.3 Linear Time Invariant

A specialized case of state space system theory, and probably the most commonly used, is one for which a system obeys the time shifting property. Suppose a system of the form of Eq. (11) has an input \( u \) and a response that are smooth \( C^1 \) vector valued functions defined for time \( t \geq t_0 + \tau \) for \( t_0 \) and \( \tau \in \mathbb{R} \). Then the system is time invariant if time shifting the initial state to \( t_0 + \tau \) while applying the same input from \( t_0 + \tau \) results in the same response [37].

\[
\begin{align*}
&x(t_0 + \tau) \\
u(t - \tau)
\end{align*}
\rightarrow y(t - \tau)
\]  

(21)

Time invariance implies that the coefficient matrices of Eqs. (11) and (16) are constant with time, and therefore the STM simplifies to the matrix exponential of the system matrix.
\( \Phi(t,t_0) = e^{A(t-t_0)} \). This is the typical linear time invariant system that results from linearizing a nonlinear system about a single equilibrium state. In this dissertation, a linearization is applied about all instants in a periodic orbit, which results in a linear time periodic system. Then, a set of methods are used which transform the linear time periodic response into the mathematical form of a linear time invariant system. For example, suppose that the time shift was an integer multiple of the fundamental period \( \tau = T \). Then the linear time periodic system would seem to be linear time invariant and could be treated using familiar theory for LTI systems. (In essence this is exactly what is done in Floquet theory to evaluate the stability of the LTP system). So the linear time invariant class of systems is important for this dissertation as well.

### 2.5 Remarks

One of the most important concepts of the systems theory presented in this section is that a nonlinear system can be linearized about a single equilibrium state which gives rise to linear time invariant system dynamics. Alternatively, a nonlinear system can be linearized about a whole periodic orbit, which results in a system that is well approximated as linear time periodic. This is the key point that is exploited for the identification procedure in this dissertation. Then, the remaining ideas regarding stability and the state space solutions are tools for performing the calculations.

### 3 Experimental System Identification Techniques for LTP Systems

#### 3.1 Introduction

The previous section illustrated that a wide class of nonlinear systems can be modeled as linear time periodic for small perturbations about a limit cycle. There are also many other
systems that can be modeled as LTP. For example, researchers looked at analytical models of physical systems such as rotating machinery [28, 42], wind turbines [43-45], buckling columns [46], and human gait systems [27, 47, 48].

Only recently have some techniques been suggested for identifying parametric models from measurements of linear time periodic systems. Some significant time domain techniques have been developed [49-51] that can be employed for systems with rapidly varying, periodic dynamics. The techniques were not extended to the frequency domain and the work did not include reconstruction of the time varying matrices that describe the system. Some significant analytical frequency domain techniques were proposed by Wereley [52] and Hall [24, 53]. They developed a foundation for the Harmonic Transfer Function (HTF), which is an LTP analog to the frequency response function (FRF) used for linear time invariant systems. Later, some of these methods were used by Siddiqui [54] and Hwang [55] to identify models from measurements of helicopter rotors.

Allen has suggested [21] two approaches that can be used to identify parametric models of time periodic systems from experimental measurements. Allen applied these techniques to the analysis of a time periodic Jeffcott rotor [21]. Allen and Sracic have applied these techniques to the application of Continuous-Scan Laser Doppler Vibrometry (CSLDV) [41, 56] and to Output-only Modal Analysis (OMA) of a simulated wind turbine [25]. As preliminary work for this dissertation, Sracic and Allen to applied these methods to identify parameters of nonlinear systems. Some of these results have been recently published [26] or submitted for publication [57]. This dissertation presents a new system identification routine to extract the parameters of the approximate linear time periodic response resulting from removing the periodic orbit from a perturbed, time periodic nonlinear system. Sections 3.2 and 3.3 will provide the suggested
formulations to extract the parameters from the approximated time periodic response. Then, Section 3.4 will provide the methods to estimate the time periodic state matrix and the model of the original nonlinear system.

3.2 Free Response Approaches

The free response of a linear time periodic system is often very difficult to interpret. The frequency domain representation of a response signal from an LTP system that is excited by impulse may contain many peaks due to harmonics from time periodic effects. These peaks can be interpreted, or collected for model extraction, using the Fourier Series Expansion (FSE) method. An alternative approach is to consider a lifting transformation that essentially resamples the full LTP response signal modulo($i2\pi/T$). The key result of either of these methods is that the resulting expression of the signal is equivalent to the expression of a system that is LTI. Then, any standard parameter extraction technique can be used to identify a model of the system. These ideas are developed in the sections here.

3.2.1 Fourier Series Expansion Technique

In Section 2.4.2, the state transition matrix was introduced, factored using Floquet theory, and shown in the diagonalizable case to be similar to the matrix exponential of the Floquet exponents multiplied by a time periodic matrix (i.e. $\Phi(t,t_0) = \Psi(t) e^{A(t-t_0)} [\Psi(t_0)]^{-1}$). The matrix $[\Psi(t)] = [\phi_1(t), \phi_2(t) \cdots \phi_n]$ contains the time varying Floquet eigenvectors $\phi_i(t) \in \mathbb{C}^n$ where $n$ is the number of states of the system. This Floquet form of the state transition matrix can be composed in summation form as
\[
\Phi(t, t_0) = \sum_{r=1}^{n} \left[ R(t) \right]_r e^{\lambda_r (t-t_0)}
\]

(22)

\[
\left[ R(t) \right]_r = \{ \phi(t) \}_r, \{ \psi(t_0) \}_r
\]

where \([R(t)]_r\) is the \(r^{th}\) residue matrix corresponding to the \(r^{th}\) Floquet exponent \(\lambda_r \in \mathbb{C}\) and is composed of the product of the \(r^{th}\) right Floquet eigenvector \(\phi_r(t)\) and the left Floquet eigenvector \(\{ \psi(t_0) \}_r \in \mathbb{C}^n\). The residue matrix \([R(t)]_r\) is periodic because it is proportional to the periodic eigenvector \(\{ \phi(t) \}_r\), so it can be expanded in a Fourier series.

\[
\left[ R(t) \right]_r = \sum_{m=-N_B}^{N_B} \left[ B_m \right] e^{i m \omega_r (t-t_0)}
\]

(23)

where \([B_m]_r\) is the \(m^{th}\) Fourier coefficient matrix of the \(r^{th}\) mode and \(\omega_r = 2\pi/T\) is the period frequency. After substituting this expression into Eq. (22) and simplifying terms, the result is the following

\[
\Phi(t, t_0) = \sum_{r=1}^{n} \sum_{m=-N_B}^{N_B} \left[ B_m \right] e^{(\lambda_r + i m \omega_r) (t-t_0)}
\]

(24)

where \([B_m]_r \in \mathbb{C}^{n \times n}\) is the \(m^{th}\) Fourier coefficient matrix of the \(r^{th}\) mode and \(\omega_r = 2\pi/T\) is the period frequency. To be exact, the Fourier expansion should include an infinite number of terms, but in practice one can truncate the series at the \(N_B^{th}\) order term as in the equation above and retain acceptable accuracy.

The FSE representation can also be developed for the system output by first expressing

\[y(t) = C(t)\Phi(t, t_0)x(t_0)\]

in summation form as
where the $r^{th}$ output residue vector $\{R_r(t)\}_r$ now incorporates the periodic output matrix $C(t)$ and the initial state $x(t_0)$ [21] and is periodic. By expanding $\{R_r(t)\}_r$ in a Fourier series, the output of Eq. (25) can be written with the same FSE representation as Eq. (24).

In Eqs. (24) and (25), each exponential term and Fourier coefficient pair $(\lambda_r + im\omega_r, [B_m]_r)$ is manifested in the spectrum of the approximate linear time periodic response as a peak similar in appearance to a linear mode peak. The response spectrum of an linear time invariant system contains one peak for each eigenvalue at $\omega \approx \text{Im}(\lambda_r)$. In contrast, Eq. (24) shows that a linear time periodic system may have additional peaks at frequencies $\text{Im}(\lambda_r + im\omega_r)$ where $m$ is any integer. In practice only some of these peaks will rise above the noise floor of the measurement. Spurious Fourier coefficients, $[B_m]_r$, sometimes have surprisingly large effect on the time periodic model (e.g. when reconstructing $A(t)$ as described in Section 3.2.3) so it is important to inspect the measurements and only include terms that are clearly meaningful. Spurious peaks do not usually have the familiar Lorentzian shape, so they can be discerned from true mode peaks by a simple visual inspection. Since there may be dozens of peaks in a spectrum, it is helpful to automate this procedure by calculating the correlation between each peak and the best single-mode approximation. Specifically, denoting the measured spectrum at the $j^{th}$ DOF as $H^{\text{meas}}_j(\omega)$ and similarly for the spectrum reconstructed by curve fitting a single mode at that peak $H^{\text{fit}}_{j,r}(\omega)$, the correlation factor is defined as follows.
The correlation, \( C_{j,r} \), for the \( r^{th} \) mode at the \( j^{th} \) DOF is a measure of the difference between the measurement and the curve fit, relative to the magnitude of the measurement. Only the values in the spectra \( \omega_k \) within the half power frequencies \( \omega_{L,r} \) and \( \omega_{H,r} \) of the peak in question are used for the calculation. Perfect correlation gives a value of 1 indicating that the curve fit to the data near the peak in question fits the measurement perfectly. This metric can be used to determine which peaks to include in the identified model. Experience has revealed that a Fourier term is sufficiently accurate to include when reconstructing \( A(t) \) if \( C_{j,r} \geq 0.8 \).

It would be quite tedious to manually curve fit each of the peaks in a spectrum, and when this approach is used one must also average the eigenvalues in some way since each gives a slightly different estimate for \( \lambda_r \). In this work the authors instead employ the lifting method, which is described in the next section, and then use a fast Fourier transform to estimate the Fourier coefficients \( \left[ B_m \right] \). The FSE method is only used to visualize the measurements and to compute the correlation metric.

### 3.2.2 Lifting Technique

When the fundamental period of the linear time periodic system response is known, one can perform an analysis based on “lifting” [21]. Using this analysis, the original linear time periodic response is resampled into a larger set of signals that are augmented in time. Let the continuous-time output response \( y(t) \in \mathbb{R}^q \) be from a periodic system with period \( T \). Let \( y_j \) denote the \( j^{th} \) sample of the response, or \( y(t_j) \). If the response has been sampled an integer \( p \)
times per period over \( N_c \) periods, then one could create a response vector that is sampled only once per fundamental period. For example, starting at the \( k^{th} \) sample, one could form \( y_{k+mp} \) for \( m \) ranging from 0 to \( N_c-1 \). One could do the same for each possible starting instant \( k = 0 \ldots p-1 \).

The lifted response vector, \( y^L \), is merely a collection of these sampled-once-per-period responses for all possible starting instants. So, the \( m^{th} \) sample of the lifted response vector is denoted \( y^L_m \in \mathbb{R}^{mp} \) and is given by the following [21].

\[
y^L_m = \begin{bmatrix} y_{0+mp}^T, y_{1+mp}^T, \cdots, y_{p-1+mp}^T \end{bmatrix}^T
\]  

(27)

If the system input is zero, then one can use Eq. (20) to show that the periodic residue vectors appear to be constant due to the lifting, and then the lifted free response reduces to

\[
y^L_m = \sum_{r=1}^{n} \left\{ R^L_{rd} \right\}_r e^{\lambda_r mT}
\]

(28)

where \( \left\{ R^L_{rd} \right\}_r \in \mathbb{R}^{mp} \) is the residue vector of size \( p \) times the number of outputs of the response, \( y \).

The \( k^{th} \) block of the residue vector has the following form.

\[
\left\{ R^L_{rd} \right\}_{k,r} = \left\{ R_y(t_k) \right\}_r e^{\lambda_r (t_k-t_0)} , \quad k = 0 \ldots p-1
\]

(29)

The exponential in this equation appropriately accounts for the time delay between the initial time \( t_0 \) and the \( k^{th} \) response vector, \( y_k \). Furthermore, if \( \left\{ R^L \right\}_r \in \mathbb{R}^{mp} \) is defined by Eq. (29), but without the exponential term, then it is just the lifted form of \( \left\{ R_y(t) \right\}_r \) the continuous-time varying residue vector from Eq. (25) [21]. The advantage of forming the lifted free response of Eq. (28) is that there is only one exponential term for each mode of the time periodic system instead of an exponential term for each mode and all of its harmonics, so the lifted response is
directly analogous to the free response of a linear time invariant system. Each exponential term therefore gives rise to a peak in the frequency response, and one can use standard linear time invariant system identification methods to identify a parametric model for the system. For example, see how Fig. 6 reduces to Fig. 7 after the lifted method is applied.

The form of Eq. (27) shows clearly how subsequent response vectors are spaced by a time increment equal to the system period \( T \). Therefore, it is important to realize that the lifting procedure changes the effective bandwidth of the lifted response is \( (0, \omega_r/2) \), which may alias the response [21, 41]. If the signal represents the free response of a system, then modes which occur at frequencies higher than \( \omega_r/2 \) will be aliased. For example, let \( \omega_n \) be a natural frequency of the system, and assume that the response has contains the mode associated with \( \omega_n \). Then, if \( \omega_n > \omega_r/2 \), the frequency will be aliased according to the following equation [41].

\[
m = \frac{\omega_n - \omega_n \pmod{\omega_r/2}}{(\omega_r/2)}
\]

\[
\omega_{n,\text{aliased}} = \begin{cases} 
-1^{(m)} \omega_n & m = 0 \\
-1^{(m)} \omega_n + (-1)^{(m+1)} \frac{m+1}{2} \frac{\omega_r}{2}, & m \text{ odd} \\
-1^{(m)} \omega_n + (-1)^{(m+1)} \frac{m}{2} \frac{\omega_r}{2}, & m \text{ even}
\end{cases}
\]

The term ‘\( m \)’ in the equation is calculated using the modulo function [58] which gives the remainder of the first term divided by the second term. Specifically, \( a \pmod{b} \) is the remainder after \( a \) is divided by \( b \). For example \( 22 \pmod{10} = 2 \), \( 6 \pmod{3} = 0 \), and \( 3 \pmod{7} = 3 \). The effect of aliasing is significant because the free response contains a linear superposition of all the
modes contributing to the response. One often has control over the frequency of the periodic system so one can avoid aliasing by choosing the frequency \( \omega_T \) so that the effective bandwidth \( \omega_T/2 \) is sufficiently high. Alternatively, the aliased modes provide a perfectly valid representation for the linear time periodic system model, so the previous equation may simply be used to understand the true frequencies of the system's modes.

### 3.2.3 Constructing the State Vector Model

In [21], Allen discusses how the state transition matrix can be constructed from either of the models above assuming that one has measured only the displacement of the system. The velocities needed to complete the lower half of the state vector were obtained by differentiating the Fourier series model for the displacements. For many systems it is more convenient to measure acceleration, especially for structural dynamic systems. When this is the case, the Fourier series expansion model that one identifies using the proposed methodology corresponds to the acceleration of the system, denoted \( x_a(t) \in \mathbb{R}^N \). The identified model can be written as

\[
x_a = \sum_{r=1}^{n} \sum_{m=-N_B}^{N_B} \left\{ B_m \right\}_r e^{(\lambda_r + im\omega_T)(t-t_0)}
\]

(31)

where \( N \) is the number of degrees of freedom and \( \left\{ B_m \right\}_r \) is the residue vector. In order to create a full state vector that consists of \( N \) positions \( x_d(t) \in \mathbb{R}^N \) and \( N \) velocities \( x_v(t) \in \mathbb{R}^N \), one can integrate the previous equation two times successively.

\[
x_v = \sum_{r=1}^{n} \sum_{m=-N_B}^{N_B} \left\{ B_m \right\}_r \left( \lambda_r + im\omega_T \right)^{-1} e^{(\lambda_r + im\omega_T)(t-t_0)}
\]

\[
x_d = \sum_{r=1}^{n} \sum_{m=-N_B}^{N_B} \left\{ B_m \right\}_r \left( \lambda_r + im\omega_T \right)^{-2} e^{(\lambda_r + im\omega_T)(t-t_0)}
\]

(32)
Then, the state transition matrix and state coefficient matrix can be calculated as in [21]. There are $n^2$ unknowns in the state coefficient matrix at each time step, and the Fourier series expansion model can be used to form a system of equations to solve for these unknowns. The previous equation provides the Fourier series expansion model for the full state vector $x(t)^T = [x_d(t)^T, x_r(t)^T]$ due to initial conditions $x(t_0)$. Additional responses can be formed by shifting the the model by $mT$ where $m$ is an integer. Since the Fourier series expansion model is time periodic, $x(t + mT)$ is equal to $x(t)$, and each additional shifted response is generated by multiplying the Fourier series expansion model by $e^{\lambda_r mT}$.

$$x(t + mT) = \begin{bmatrix} x_d(t) \\ x_r(t) \end{bmatrix} e^{\lambda_r mT}, \quad m = \text{integer}$$  \hspace{1cm} (33)

Then, using the free response form of Eq. (12), $x(t) = \Phi(t,t_0)x(t_0)$, and the shifted responses, the following system of equations can be formed.

$$[x(t) \cdots x(t + (m-1)T)] = \Phi(t,t_0)[x(t_0) \cdots x(t_0 + (m-1)T)]$$  \hspace{1cm} (34)

Following [21], the constant matrix of initial conditions on the far right hand side of the previous equation is denoted $[X_{IC}]$. This constant matrix can be inverted to solve for the state transition matrix in the previous equation, and Eqs. (32) and (33) can be inserted, which gives the following equation.

$$\Phi(t,t_0) = \sum_{r=1}^{n} \sum_{m=-N_B}^{N_B} \left\{ \begin{bmatrix} B_m \end{bmatrix}_r \left( \lambda_r + im\omega_r \right)^{-1} \right\} \left[1 \cdots e^{\lambda_r nT} \right] [X_{IC}]^{-1} e^{(\lambda_r + im\omega_r)(t-t_0)} \hspace{1cm} (35)$$

After forming the state transition matrix, Eq. (14) can be rearranged to solve for the state coefficient matrix.
\[ A(t) = \left( \frac{d}{dt} \Phi(t, t_0) \right) \Phi(t, t_0)^{-1} \]  

(36)

The time derivative of the state transition matrix can be formed by numerically differentiating Eq. (35) at each time step in the measured response, or alternatively the state transition matrix could be formed a second time using the velocity and acceleration expressions from Eqs. (31) and (32).

### 3.2.4 Discussion of Free Response Approaches

Both the Fourier series expansion and lifting approaches provide a useful tool for extracting a model of the approximate linear time periodic response. It is often more convenient to use the lifting technique to extract a set of time periodic modes from the response because the Fourier series expansion method requires a bit of user intervention. Once \( \{ \hat{R}^{ld} \}_r \) and \( \lambda_r \) have been obtained from the lifting method, one can always use the discrete Fourier Transform to compute the Fourier coefficients, \( [B_m]_r \), of a Fourier series model from \( \{ \hat{R}^{ld} \}_r \). This will be illustrated in Section 4. Once the Fourier coefficients are known, the state transition matrix can be reconstructed using Eq. (35) and the state coefficient matrix, \( A(t) \), can be reconstructed using Eq. (36).

### 3.3 Forced Response Approaches

In many applications, engineers are primarily concerned with specific operating conditions of a structure. In this case, the system is not responding freely, but more generally is being actively forced or is experiencing forcing due to environmental conditions. Performing analysis on systems responding in these conditions when the force is not measured is often called
Operational Modal Analysis (OMA) or Output-only Modal Analysis. The topic has been heavily studied since its origins, and while many papers consider the techniques for linear time invariant systems or simple linear time varying systems, only a few groups have applied to methods to linear time periodic systems. The techniques of Werely, Hall, and Sidiqui can be used to develop an operation modal analysis technique for time periodic models, which was explored for simulated measurements of a wind turbine in [59] and measurements of a 20kW wind turbine in a paper that was recently submitted for publication [60]. This technique might also be used to approximate the forced response of a periodic nonlinear system. The suggested operational modal analysis technique for time periodic systems is summarized here.

3.3.1 Output-only Modal Analysis for Time Periodic Systems

Let \( y(t) \in \mathbb{R}^q \) be the output of a linear time varying system. Suppose that the operating condition is one where sufficient time has allowed for the homogeneous portion of the solution to decay.

\[
y(t) = C(t) \int_0^t \Phi(t, \tau) B(\tau) u(\tau) d\tau + D(t) u(t) \quad (37)
\]

The resulting response is then composed of \( q \) outputs of the \( n \)-state system. If the system is periodic and has a Floquet representation, then it has periodic residue vectors that can be expanded in a Fourier series. Using the Floquet representation of the STM from Eq. (24), the left side and right side expansions of the periodic residues are given by

\[
C(t) \{ \phi(t) \}_r = \sum_{m=-\infty}^{\infty} \left\{ \overline{C}_m \right\}_r e^{im\omega_r t} , \quad \{ \psi(t) \}_r^T B(t) = \sum_{m=-\infty}^{\infty} \left\{ \overline{B}_m \right\}_r e^{im\omega_r t} \quad (38)
\]
where \( \{ \mathbf{C}_m \}_r \subset \mathbb{C}^q \) and \( \{ \mathbf{B}_m \}_r \subset \mathbb{C}^{1 \times p} \) are the \( m^{th} \) Fourier coefficient vectors of the \( r^{th} \) mode of \( C(t) \{ \phi(t) \}_r \) and \( \{ \psi(t) \}_r^T B(t) \), respectively.

Commonly, a harmonic response in the form of Eq. (37) is achieved for a linear time invariant system when one excites the system with a single frequency harmonic forcing function. The system will then respond harmonically at the same frequency but with different phase and amplitude. When a linear time periodic system is excited with a single frequency harmonic forcing function, it may respond at an infinite number of frequencies, making the typical definition of the transfer function inapplicable. One can circumvent this issue by using an exponentially modulated periodic (EMP) input signal space [53-55] in which the input is modulated with a complex exponential so that it provides energy at an infinite number of frequencies.

\[
u_{EMP}(t) = \sum_{m=-\infty}^{\infty} u_m e^{(i\omega_m t)}
\]  
(39)

The input \( \hat{u}(t) \) will produce a harmonic output sharing the same infinite frequency space.

\[
y_{EMP}(t) = \sum_{m=-\infty}^{\infty} y_m e^{(i\omega_m t)}
\]  
(40)

The vectors of harmonic magnitudes \( u_m \) and \( y_m \) for the \( m^{th} \) harmonic of the input and output respectively can be used to create a relationship for exponentially modulated periodic input-output signals that is analogous to the linear time invariant transfer function, and this relationship has been referred to as the Harmonic Transfer Function (HTF) [52].

\[
\hat{Y}(\omega) = \hat{G}(\omega)\hat{U}(\omega)
\]  
(41)
A summary of the technique starts by taking the measured response of the system and subsequently multiplying it by \( e^{im_0 \omega t} \) to get the desired harmonic in the expansion of Eq. (40). Then, the shifted harmonic responses, \( y_m \), are collected to form the following response.

\[
\hat{y}(t) = \begin{bmatrix}
    \cdots & y_{-2}^T & y_{-1}^T & y_0^T & y_1^T & y_2^T & \cdots
\end{bmatrix}^T
\]  

(42)

The Fourier transform of each shifted response can also be computed,

\[
Y_m(\omega) = \int_{-\infty}^{\infty} y(t)e^{(-i\omega - im_0 \omega t)} dt
\]  

(43)

and all of the shifted Fourier transformed responses can be collected to form the following response in the frequency domain.

\[
\hat{Y}(\omega) = \begin{bmatrix}
    \cdots & Y_{-1}(\omega) & Y_0(\omega) & Y_1(\omega) & \cdots
\end{bmatrix}^T
\]  

(44)

The measured input, \( u(t) \) can be treated in a similar fashion. First, the harmonic shifted copies are formed by multiplying by \( e^{im_0 \omega t} \), then the Fourier transforms are computed, and finally the matrix \( \hat{U}(\omega) \) of shifted frequency domain responses is formed. Both of these vectors are infinite in size because they are made up of an infinite number of the coefficient vectors in Eqs. (39) and (40). In practice though, a finite number of the coefficient vectors will adequately describe the EMP signal spaces. The task remains to generate a function \( \hat{G}(\omega) \) that relates the input and output EMP spaces that retains analogous characteristics of the transfer function for linear time invariant systems. Wereley presents a full derivation of the HTF in his thesis [52] and the result is provided here.
\[
\hat{G}(\omega) = \sum_{r=1}^{n} \sum_{l=-\infty}^{\infty} \frac{\{\hat{C}_l\}_r \{\hat{B}_l\}_r}{i\omega - (\lambda_r - il\omega_T)} + \hat{D}
\]

\[
\{\hat{C}_l\}_r = \begin{bmatrix} \cdots \{\overline{C}_{l-1}\}_r^T \{\overline{C}_{l-1}\}_r^T \{\overline{C}_{l-1}\}_r^T \cdots \end{bmatrix}^T
\]

\[
\{\hat{B}_l\}_r = \begin{bmatrix} \cdots \{\overline{B}_{l+1}\}_r \{\overline{B}_l\}_r \{\overline{B}_{l-1}\}_r \cdots \end{bmatrix}
\]

In Eq. (45), the \(j^{th}\) term in the \(\{\hat{C}_l\}_r\) is \(\{\overline{C}_{j-l}\}_r\), which is the \((j-l)^{th}\) Fourier coefficient of \(C(t)\{\phi(t)\}_r\), the \(k^{th}\) term in the vector \(\{\hat{B}_l\}_r\) is \(\{\overline{B}_{l-k}\}_r\), which is the \((l-k)^{th}\) Fourier coefficient of \(\psi(t)_r^T B(t)\), and the \((j,k)^{th}\) element of \(\hat{D}\) is the \((j-k)^{th}\) Fourier coefficient of \(D(t)\). Eq. (45) has the same mathematical form of the Frequency Response Function (FRF) for a linear time invariant system that is composed of the modal parameters of the system. This is greatly beneficial for the analysis of linear time periodic systems because it means that the same algorithms and techniques used for the well known FRF can be used for the HTF.

There are also some differences between the frequency response function and the harmonic transfer function that will be very important for the correct interpretation of HTFs. First, unlike linear time invariant systems that have as many peaks in their FRFs as the order of the system, a linear time periodic system of order \(n\) can potentially have an infinite number of peaks in its HTF, depending on how many terms in \(\{\hat{C}_l\}_r\) and \(\{\hat{B}_l\}_r\) are nonzero. The peaks occur near the imaginary part of the Floquet eigenvalue \(\lambda_r\) plus some integer multiple of \(\omega_T\), or essentially when the denominator of Eq. (45) tends to zero. The second important difference is that the mode vectors \(\{\hat{C}_l\}_r\) of an LTP system are not just a representation of spatial deformation of the system, as in the case of an LTI system. The mode vectors have components that are
Fourier coefficients for each state component that describes how that state component is changing with time.

These techniques for forced response approaches are not used in this dissertation. However, they provide a promising outlook for applying the NL-LTP approach to perturbation responses that include a perturbation force.

### 3.4 Techniques for Estimating the Nonlinear System Model

Once a time periodic model has been estimated for the perturbation about the nonlinear periodic orbit, it must be used to estimate the original nonlinear system. This section provides the necessary techniques to estimate the nonlinear equations of motion. Additionally, the methods of this section can be used to gain further insight into the identified model if the system of interest is a structural dynamic system. For example, the individual nodal restoring accelerations and the net nodal restoring forces can both be estimated with the proposed methods.

#### 3.4.1 Estimating the Nonlinear Equations of Motion

In Eq. (10), the original nonlinear system model was differentiated to obtain the time varying coefficient matrices of the linear time periodic model, which resulted in a total derivative form for the system.

\[
df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial u} du
\]

It is assumed that the coefficient matrix \( \frac{\partial f_i}{\partial x_j} \) depends only on \( x \) and \( \frac{\partial f_i}{\partial u_j} \) only on \( u \), which is an accurate assumption for structural dynamic systems since the input and state
contributions of the equations of motion are rarely represented as product combinations (i.e. \(ux, ux^2\), etc.). When this assumption holds, each component in \(A(t) = \left[ \frac{\partial f_i}{\partial x_j} \right]_{\vec{x}(t)} \) can be integrated (recall that function \(f\) was assumed to be at least piecewise \(C^1\) in order to calculate \( \left[ \frac{\partial f_i}{\partial x_j} \right] \), so integration is valid) with respect to the corresponding component of \(\vec{x}(t)\) to obtain a representation as follows

\[
f_{i,j}^{x} (x_j) = \int_{\vec{x}^{(0)}}^{\vec{x}^{(T)}} A_{(i,j)} dx_j + C_{i,j}^{x}, \quad i, j = 1, \ldots, n, \quad n = 2N
\]  

(47)

where \(f_{i,j}^{x} (x_j)\) is the integral of the component in the \(i^{th}\) row and \(j^{th}\) column of \(A(t)\) and the integral is taken with respect to the \(j^{th}\) component of the state vector along the periodic orbit \(\vec{x}(t)\). The term \(C_{i,j}^{x}\) is a constant of integration that accounts for nonzero initial conditions at \(\vec{x}^{(0)}\). This gives only one component of the matrix \(\left[ f^{x}(x) \right] \in \mathbb{R}^{n \times n}\). The total function \(f^{x}(x) \in \mathbb{R}^{n}\) can be found by adding up all of the components \(f_{i,j}^{x} (x_j)\) for each state \(x_j\),

\[
\hat{f}^{x}(x) = \sum_{j=1}^{n} \left\{ f_{j}^{x}(x_j) \right\}
\]  

(48)

where \(\left\{ f_{j}^{x}(x_j) \right\}\) is the \(j^{th}\) column of \(\left[ f^{x}(x) \right] \). If the \(B(t) = \left[ \frac{\partial f_i}{\partial u_j} \right] \) matrix has been identified or is known, it could also be treated in a similar manner to obtain \(\hat{f}^{u}(u)\). Then the following equation defines the reconstructed nonlinear equations of motion.

\[
\hat{f}(x,u) = \hat{f}^{x}(x) + \hat{f}^{u}(u)
\]  

(49)
3.4.1.1 Example:

To illustrate this procedure, consider a simple system whose state is the velocity of a particle of unit mass that is subjected to a unit aerodynamic drag force and an applied force \( u \), giving the equation of motion \( \dot{x}_i = -x_i^2 + u \). Suppose that \( u = \bar{u} \) is chosen to cause the system to oscillate in a periodic orbit \( \bar{x}_i = a \cos(\omega_t t) \). The linearized system is \( \dot{\hat{x}}_i = A(t)\hat{x}_i \) with \( A(t) = -2\bar{x}_i = -2a \cos(\omega_t t) \). (One could verify that integrating \( A(t) = -2\bar{x}_i \) as explained above recovers the correct term in the nonlinear equation of motion: \( \dot{f}^x(x_i) = -x_i^2 \).) Using the system identification procedure described in the preceding sections, one would measure the periodic orbit \( \bar{x}_i{}^{\text{est}} \), perturb the system from that orbit and measure \( \hat{x}_i \) (or alternatively \( \{\hat{y}\} \) as described in [21]), and use that to estimate \( A(t) \). Those identified quantities could be expressed in a number of ways, one of the most convenient being a vector of samples describing the estimated state matrix, \( A(t)^{\text{est}} \), over the periodic orbit \( 0 < t < T \). When that is the case, the integral in Eq. (47) can be evaluated numerically using the Trapezoid rule with \( \bar{x}_i{}^{\text{est}} \) as the ordinate and \( A(t)^{\text{est}} \) the abscissa to recover a sampled estimate of \( \dot{f}^x_{1,1}(x_i) \) over the range of \( \bar{x}_i{}^{\text{est}} \).

3.4.2 Applications to Structural Dynamic Systems

3.4.2.1 System Restoring Forces

A broad range of structural dynamic systems can modeled with the following equations of motion,

\[
M \dot{x}_a + G_{RF} (x_d, x_v) = F_{\text{ext}} (t)
\]  

(50)
where $M$ is the mass matrix, $G_{RF}(x_d, x_v)$ is vector that gives the net restoring forces that act on each degree of freedom as a function of the displacements and velocities of each degree of freedom, and $F_{ext}(t)$ is the vector of applied forces. This system can be written in state space with the state vector $x(t)^T = [x_d(t)^T, x_v(t)^T]$.

$$
\begin{bmatrix}
  x_v \\
  x_a
\end{bmatrix} = \begin{bmatrix}
  x_v \\
  -g_{RF}(x_d, x_v) + M^{-1}F_{ext}(t)
\end{bmatrix},
$$

$$
g_{RF}(x_d, x_v) = M^{-1}G_{RF}(x_d, x_v)
$$

In the equation, the function $g_{RF}$ gives the accelerations of each degree of freedom in the system, which are nonlinear functions of the states. When this system is linearized about a periodic trajectory, the time periodic state space equations take the following form.

$$
\begin{bmatrix}
  \ddot{x}_v \\
  \ddot{x}_a
\end{bmatrix} = \begin{bmatrix}
  0 & I \\
  -\frac{\partial g_{RF}}{\partial x_d} & -\frac{\partial g_{RF}}{\partial x_v} & \frac{\partial F_{ext}}{\partial x_d} & \frac{\partial F_{ext}}{\partial x_v}
\end{bmatrix} \begin{bmatrix}
  \dot{x}_d \\
  \dot{x}_v
\end{bmatrix} + \begin{bmatrix}
  0 \\
  M^{-1} \frac{\partial F_{ext}}{\partial x_d} + \frac{\partial F_{ext}}{\partial x_v} F_{ext}(t)
\end{bmatrix}
$$

(52)

The terms in this equation show that the terms in the first $N$ rows of $f^x(x)$ simply provide an identity relationship that causes the velocities to be the derivatives of the displacement, so we turn our attention to the terms in the lower $N$ rows. Let $g^x(x)$ be the matrix that describes the lower $N$ rows of $f^x(x)$.

$$
[g^x(x)] = [f^x_{N+1:N,1:N}(x)]
$$

(53)

The components of $g^x(x)$ are the accelerations in the system. In particular, $g^x_{i,j}(x_j)$ is the nonlinear acceleration of the $i^{th}$ DOF caused by the $j^{th}$ component of the state vector.
Furthermore, the form of Eq. (52) shows that the components of \( g^x(x) \) are essentially a nonlinear force-displacement relationship if \( x_j \) is a displacement and a nonlinear force-velocity relationship if \( x_j \) is a velocity, although all the components are scaled by mass. Hence, this method allows one to determine the individual contributions of each state to the restoring forces. Each of these relationships still contains an unknown constant of integration, but those constants, which comprise the constant matrix \( C^x \), can be determined if one knows the equilibrium point (typically \( x=0 \)) for the restoring accelerations; one simply has to shift the acceleration-displacement or acceleration-velocity curves until they pass through that point. The net acceleration of each degree of freedom is then calculated with the following

\[
\{ g(x_d, x_v) \} = \sum_{j=1}^{n} \{ g_j^x(x_j) \} \tag{54}
\]

where \( \{ g_j^x \} \) is the \( j^{th} \) column of \( g^x(x) \).

### 3.4.2.2 Modified Restoring Force Surface Method for Periodic Response

So far the periodic orbit has only been used to describe the state at which the linearized system exists, but \( \bar{x}(t) \) itself may contain valuable information regarding the nonlinearity in the system. This section presents a method based on the restoring force surface approach [7] that uses \( \bar{x}(t) \) to compute the net restoring accelerations in the system over the periodic orbit. This estimate can then be compared to the model in Eq. (47) in order to validate that result.

The restoring force surface method is based on Eq. (50). For a single degree of freedom system one can simply rearrange the equation to estimate \( G_{RF}(x_d, x_v) \) given measurements of \( x_a \) and \( F_{ext}(t) \), and then plot \( G_{RF}(x_d, x_v) \) versus \( x_d \) and \( x_v \) to understand how the restoring forces
depend on the states. The method is elegant for single-degree-of-freedom systems but becomes cumbersome for multi-degree-of-freedom systems because the effects of the various degrees of freedom are difficult to separate. Here we shall modify the traditional approach and seek only to estimate the total restoring forces in the system over the periodic orbit. Assuming that the acceleration of each of the nodes of the system as well as the applied force $F_{\text{ext}}(t)$ have been measured, the total restoring force can be found as follows if one has an estimate for the mass matrix, $M$.

$$G_{\text{MRF}}(x_d, x_v) = F_{\text{ext}}(t) - M x_a$$  \hspace{1cm} (55)

Alternatively, the restoring accelerations can be found with the following equation,

$$g_{\text{MRF}}(x_d, x_v) = M^{-1}G_{\text{MRF}} = M^{-1}F_{\text{ext}}(t) - x_a$$  \hspace{1cm} (56)

The total restoring force changes during the periodic orbit and can be plotted versus time and compared with the restoring forces computed by the NL-LTP method.

Either of Eqs. (55) or (56) can be used to verify the results of the nonlinear identification, but this requires an accurate test analysis model (TAM) mass matrix [61]. A TAM can be derived analytically if one has a computational model of the system, or it can be found experimentally using linear modal analysis. The usual approach is to perform a traditional modal test with low amplitude hammer input and extract a set of linear modes. The mass normalized mode shape matrix that is identified experimentally, $[\phi_e]$, is an $N_m \times N$ matrix, where $N_m$ is the number of measurement points. If an adequate set of measurement points has been used (see [62] for a discussion of sensor placement strategies for linear systems), then a subset of them can
be used as states and the mass matrix can be found as follows, where the subscript \( (N_{\times N}) \) is used to stress the fact that the appropriate square partition of \( [\phi_e] \) should be used in the computation.

\[
[M_e] = ([\phi_e]_{N_{\times N}})^{-T} ([\phi_e]_{N_{\times N}})^{-1}
\]  

(57)

3.5 Summary

Using the NL-LTP approach, one must first approximate the system as linear time periodic and identify the time periodic parameters, then these can be used to reconstruct the nonlinearities in the original responses. The methods in this section provide the mathematical foundations to extract the time periodic modes, estimate the time periodic system model, and estimate the nonlinear system model. Additionally, the well established restoring force surface method is used to verify the model that results from the NL-LTP approach. All of the necessary theory has been established and a verification technique has also been suggested. The NL-LTP technique will now be applied to several systems to evaluate its performance. The first system considered is the well known Duffing oscillator.

4 Application to a Duffing Oscillator

4.1 Introduction

All of the theory developed in the previous sections is applicable to systems of arbitrary order. As with linear time invariant systems, additional modes appear in the response as the system order is increased, but otherwise the procedure does not change. This section demonstrates the NL-LTP methodology on a simple single degree of freedom system, shown in Fig. 3. In particular, the Duffing oscillator is a classic example that has been used by many
researchers to explore newly developed identification methods. The system can be solved with many analytical and numerical methods, so the results of the identification can be compared to solutions from widely known methods. Additionally, the Duffing oscillator can be used to introduce the types of complex phenomena that are often displayed by common nonlinear systems. This section will therefore begin with a primary example where the identification is applied to the Duffing system. Then, a brief introduction to bifurcation theory is presented, and the bifurcations of the Duffing system, which lead to many possible periodic orbits, are explored. The identification is then applied to the system when a number of different forcing configurations are used. This section will prepare the reader to understand the concepts of the identification procedure so that it can then be used to identify multiple-degree-of-freedom systems.

4.2 The Duffing Oscillator

Figure 3 illustrates a single degree-of-freedom (DOF) Duffing oscillator. The system has a discrete mass $m$ with displacement DOF $x_d$. The system mass is connected to a dashpot with damping coefficient $c$ and to a linear spring with spring stiffness $k$ and a nonlinear spring with spring constant $k_3$. External forcing is applied with $F_{ext}$. 
The nonlinear spring provides quadratic nonlinear spring stiffness $k_3x_d^2$, and the total restoring force of the springs is given by the following.

$$F_{sp} = kx_d + k_3x_d^3$$ (58)

Let the velocity and the acceleration of the mass be defined by $x_v(t)$ and $x_a(t)$, respectively. Then, the equation of motion is given by

$$mx_a + cx_v + kx_d + k_3x_d^3 = F_{ext}$$ (59)

which becomes the following in state space form after dividing through the entire equation by $m$ and defining the parameters $2\zeta\omega = c/m$, $\omega^2 = k/m$, $\omega_3^2 = k_3/m$, and $F = F_{ext}/m$ where $\zeta$ is the coefficient of critical damping.

$$\begin{cases} x_v \\ x_a \end{cases} = \begin{cases} x_v \\ -2\zeta\omega x_v - \omega^2x_d - \omega_3^2x_d^3 + F \end{cases}$$ (60)

The parameters used in this study are: $\zeta=0.01$, $\omega=1$, and $\omega_3=0.5$. The system is driven with the following harmonic forcing function.
\[ F(t) = A_{ext} \sin(\omega_f t) + F_d(t) \]  

where \( A_{ext} \) is the dimensionless forcing amplitude and \( \omega_f \) is the circular driving frequency. The impulsive force \( F_d \) is included to provide a disturbance to system.

The Duffing oscillator system will be used to illustrate the NL-LTP procedure:

1. The system is driven in a stable periodic orbit.
2. The disturbance force \( F_d \) is used to perturb the response from its periodic orbit. A number of cycles of this response are recorded, until the response has recovered to its original periodic orbit.
3. The underlying periodic orbit is subtracted from the response.
4. The lifting procedure is applied to the signal, and the resulting lifted responses are transformed to the frequency domain.
5. A global single DOF modal extraction program is used to fit modes to the spectrum of the lifted signals.
6. The identified modes are expanded in a Fourier series and then used to construct the state transition matrix and the state coefficient matrix of the linear time periodic (linearized) system model.
7. The state coefficient matrix is integrated to reconstruct the nonlinear restoring force of the system, which is validated using the modified restoring force method.

### 4.3 Identification Case 1

To apply the method, one must first choose what frequency and amplitude to use to produce the periodic orbit. It is desirable to excite the system so that the nonlinearity is active, so an intuitive approach is to excite the system near a resonance condition, which will occur
when the forcing frequency is near the linear natural frequency. In order to simulate the periodic response measurement, the equations of motion were integrated for an extended window equal to some large integer times the periodic of the forcing frequency until the system damping reduced the response to steady state. Then, the end conditions of limit cycle provide the initial state, $\bar{x}$, used to integrate and find the periodic orbit without any initial transient vibration. Using this approach, the initial conditions were found to be $[\bar{x}_d, \bar{x}_r]^T = [-0.0069, 0.8460]^T$ for a forcing frequency of $\omega_r = 1.0359$ rad/s and amplitude $A_{ext}=0.05$, dimensionless. One can verify that these conditions are from a stable periodic orbit by integrating the equations of motion to show that $\bar{x}(T) = \bar{x}(0)$. The Monodromy matrix [40] of the system was calculated about this periodic orbit and its eigenvalues were used to calculate the Floquet exponents, $\lambda_{an}=-0.01 \pm 0.0620i$, and the natural frequency, $|\lambda_{an}|=0.0628$ rad/s, of the periodic orbit. The damping coefficient of the time periodic mode is 0.1593. The exponents have negative real parts and therefore the periodic orbit is stable [34, 35, 40].

Using the stable periodic orbit conditions, Eqs. (60) and (61) were used to simulate the nonlinear response of the Duffing oscillator using MATLAB’s Runge-Kutta (ode45) direct time integration routine. The first integration was applied with only the harmonic forcing ($F_d=0$), and was used to simulate the periodic response $\bar{x}_d(t)$. The solution was evaluated with a sampling frequency of approximately $f_{samp}=5.28$ Hz, which results in 32 samples per cycle of the harmonic response. A second integration was then performed with $F_d$ applied in addition to the harmonic forcing to perturb the system from its steady state trajectory. A half-sine was used for the disturbance force,
\[ F_d(t) = A_d \frac{2\tau}{\pi} \sin \left( \frac{\pi}{\tau} t \right) \left( h(t_d) - h(\tau + t_d) \right) \]  

(62)

where \( A_d \) is the dimensionless amplitude of the pulse and \( \tau \) is the duration, with \( A_d = 2000 \) and \( \tau = 0.01 \) seconds. The Heaviside step functions \( h(t) \) cause the impulsive force to initiate at \( t_d \) and to disappear after \( \tau + t_d \). Both the perturbed response and the periodic response were evaluated for a time window containing 204 full cycles of the harmonic response frequency. This allowed sufficient time for the impulse response to decay to a negligible value, leaving only the steady state response \( \bar{x}_d(t) \). Figure 4 shows the time responses of the simulation. The top pane (a) contains the initial cycle of the periodic response, \( \bar{x}_d(t) \), (solid blue with markers) and the periodic response plus the perturbation, \( x_d(t) = \bar{x}_d(t) + \tilde{x}_d(t) \), (solid red). The lower pane (b) contains the difference signal, \( \tilde{x}_d(t) \), found by subtracting these two signals. The perturbation is clearly small relative to \( \bar{x}_d(t) \) and decays to essentially zero by the end of the time window.
Figure 4: Response of the system when excited by both harmonic ($\omega_f=1.0359$ rad/s and $A_{ext}=0.05$) and impulsive excitation. Plot (a) provides the initial cycle of the perturbed response (red line) and the periodic response (blue with dots). Plot (c) is the approximate linear time periodic response found by subtracting the two signals in (a).

Figures 5 and 6 contain the spectra of the responses, with Fig. 6 showing an expanded view of the low frequency band from 0 to 4 rad/s. The solid red curve corresponds to the Fast Fourier transform (FFT) of the nonlinear response, $x_d(t)$, and the dashed black curve is the spectrum of the linear time periodic response, $\bar{x}_d(t)$, which is the difference between $x_d(t)$ and the periodic trajectory $\bar{x}_d(t)$. 
Figure 5: Frequency spectra of the response of the Duffing oscillator with $\omega_r = 1.0359 \text{ rad/s}$ and $A_{\text{ext}} = 0.05$. The solid red curve corresponds to the full nonlinear response. The dashed black curve is the frequency spectrum of the linear time periodic signal, which is the difference between the nonlinear response and the periodic trajectory.

Figure 6: Plot of the same frequency spectra as Fig. 5 focusing on the low frequency range 0 to 4 rad/s.

All of the peaks in the spectra can be readily interpreted in light of the linear time periodic system theory presented in Section 2.4.2. The nonlinear response contains a dominant sharp peak at the frequency of the periodic orbit, 1.04 rad/s, as well as several others at the odd
harmonics of the orbit frequency, 3.12, 5.2, 7.28, and 9.36 rad/s. These peaks manifest the nonlinear periodic response \( \ddot{x}_d(t) \), and disappear when the linear time periodic response is found using \( \ddot{x}_d(t) = x_d(t) - \bar{x}_d(t) \). Since the periodic response only appears at a discrete set of frequencies, one could skip this step and simply ignore all of the frequencies where \( \bar{x}_d(t) \) is dominant. In any event, the rest of the spectrum contains several other peaks that have the usual shape of linear modes, for example at 0.98, 1.1, 3.06, and 3.18 rad/s. These peaks are all separated by integer multiples of the drive frequency, \( \omega_r = 1.0359 \) rad/s, (the negative frequencies -3.06 and -0.98 Hz reflect to positive frequencies), so according to Eq. (24), all of these peaks are manifestations of the same linear time periodic mode. No other peaks are visible in the spectrum, so one can surmise that the system responsible for the response \( \ddot{x}_d(t) \) has one degree of freedom and is significantly time periodic. Since each of the peaks has the distinct shape of a linear mode, there is some assurance that the observed effects are evidence of time periodicity and not just noise or some other anomaly.

Further insight can be obtained by applying the lifting technique to the linear time periodic response. This splits the time signal \( \ddot{x}_d(t) \) into 32 pseudo-linear time invariant responses since there were 32 samples per cycle of the periodic orbit, and each of the pseudo-responses has one sample per period. Figure 7 below shows a composite FFT (or average of the magnitude) of the 32 pseudo-responses (solid black line). The composite FFT contains one strong peak at 0.06 rad/s, revealing that there is only one strongly excited mode in this linear time periodic response. The Algorithm of Mode Isolation [22] was used to identify that mode’s parameters from the lifted response. The composite FFT was reconstructed using the modal parameters identified by AMI and is shown in Fig. 7 with a dashed red line. The dash-dot gray
curve corresponds to the difference between the measured response and the reconstruction, revealing that the one-mode reconstruction captures the response very well. AMI identified an eigenvalue of $\lambda_1 = -0.0097 + 0.0617i$ (the Floquet exponent), whose corresponding natural frequency $|\lambda_1| = 0.0624$ rad/s and a damping ratio, $\text{Re}(-\lambda_1)/|\lambda_1| = 0.1557$, are within 1% and 3%, respectively, of the analytical values.

The AMI routine estimates the residue vector, $\{R^{id}\}_r$ in Eq. (28), from the measurements. This is related to the periodic mode shape over one cycle, as explained in Section 3.2.2. The methods in [21] can now be used to compute the state transition matrix and the state coefficient matrix. The first step in that procedure is to expand the identified linear time periodic mode vector in a Fourier series and discard any spurious high frequency terms. Figure 8 shows the amplitudes of each of the coefficients in the Fourier series expansion of the mode vector. The open blue circles are all of the coefficients in the expansion, however not all of these
coefficients are meaningful to the time periodic response. Four coefficients clearly stand above
the noise floor, which seems reasonable since there are four coherent, time periodic peaks in Fig.
6, and the red dots in Fig. 8 are the corresponding coefficients that were retained and used to
compute the state matrix, $A(t)$. So for this response, the correlation metric (i.e. Eq. (26)) was not
needed to help select the important Fourier terms.

![Figure 8: Fourier series expansion of the linear time periodic model.](image)

*Fourier series expansion of the linear time periodic model. (open blue circles) Fourier coefficients of the identified mode. (solid red circles) Dominant Fourier coefficients that were retained when creating $A(t)$ and $\Phi(t,t_0)$.*

Using this Fourier description of the identified mode, the $A(t)$ matrix was constructed
using the method described in Section 3.2.4. Based on Eq. (52), the resulting $A(t)$ is a 2×2
matrix, and the (2,1) and (2,2) terms correspond to the instantaneous stiffness and damping,
respectively, of the system at each point within the periodic orbit. These parameters are plotted
versus time in Fig. 9 with open blue circles for the stiffness term and red asterisks for the
damping term. An analytical $A(t)$ matrix was also calculated using the known equations of
motion and the (2,1) and (2,2) terms are also shown with solid blue and red lines.
Figure 9: Components of the state coefficient matrix that correspond to the instantaneous stiffness (component (2,1)) and damping (component (2,2)) of the system for an entire periodic orbit \( (\omega_r = 1.0359 \text{ rad/s and } A_{ext} = 0.05) \).

In order to see whether the results were dependent on the system identification routine that was used, the Stochastic Subspace system identification routine (specifically, the Matlab implementation that is provided with the text by Van Overschee and De Moor [23]) was also applied to the lifted response in the time domain. SSI identified the eigenvalue for the system and the lifted residue vector, \( \{R_{ld}^l\} \), in Eq. (28). These were then processed exactly as were the AMI results and used to reconstruct the state matrix. The stiffness and damping terms in \( A(t) \) obtained from the SSI results are also shown in Fig. 9, plotted with black squares and black diamonds, respectively. The values found using the frequency-domain AMI algorithm agree well with those obtained by the time-domain SSI algorithm and both agree well with the analytical results for stiffness, so for these simulated measurements either could be used. For the remaining results of this section, only models based on modes identified with AMI will be displayed. The stiffness term changes significantly with time, varying by 50 percent over the periodic orbit. The damping in this system is linear, so the analytical \( A(t) \) component for
damping is constant with a value of 0.02 (which corresponds to the nominal damping value of $\zeta=0.01$). Although the estimated model for damping agrees with the analytical model in an average sense, it is different than the analytical value by 75% at certain instants. However, the estimated state transition matrix matched the decay of the actual response well, so the error seems to arise when estimating $A(t)$ from the state transition matrix. It is important to note that these simulated measurements were theoretically noise free. The errors that have been observed arise from error in the time integration routine and the fact that the time periodic model identified from the measurements is limited to a finite number of Fourier coefficients, $N_B$, some of which may be corrupted by errors from the identification. For a perfect results, the Fourier expansion of the mode requires an infinite number of accurate Fourier terms.

Since the periodic orbit $\mathbf{\bar{x}}_d(t)$ was measured, it was used with the identified $A(t)$ matrix to plot the instantaneous stiffness in the system (coefficient $A(2,1)$) versus the displacement. Figure 10 shows a plot of the nonlinear stiffness which seems to vary quadratically with the displacement of the system by 50 percent throughout the periodic orbit. Additionally, the figure shows that the system is stiffening since the stiffness increases (i.e. the $(2,1)$ coefficient becomes more negative) with increased displacement.
Figure 10: Stiffness versus displacement identified using the NL-LTP method with AMI (circles) compared with the analytical stiffness-displacement curve (line) ($\omega_r = 1.0359 \text{ rad/s and } A_{\text{ext}} = 0.05$).

The measured periodic response was also used in Eq. (47) to calculate the nonlinear restoring accelerations that contribute to the equations of motion. This was done using MATLAB’s ‘cumtrapz’ function, which is an approximation of the cumulative integral using the trapezoidal method. As mentioned in Section 3.4.1, there is an unknown constant of integration in each term. These were accounted for by removing the mean of each term in $[f^x(x)]$, causing for example the acceleration-displacement curves to pass through the origin.

The analytical results have been provided in the previous plots to confirm the results for this system, but this is not usually available for most experimental system, so one can first verify the results based on the nonlinear restoring accelerations. Equation (54) can be used to calculate $\{g(x_d, x_v)\}$, which is the vector of total restoring accelerations in the system at each instant within the periodic response. The total restoring accelerations can also be computed using the modified restoring force method for accelerations (Eq. (56)) and the two can be compared to validate the results. The total restoring acceleration acting on a DOF is the sum of components due to the
stiffness of the system and the dissipative effects of damping. Figure 11 shows this sum with open circles plotted in (a) versus time for one cycle of the periodic orbit. The accelerations computed using Eq. (56) are shown with the solid line. The displacement is also plotted in (b) for one cycle of the periodic response. In the first half of the cycle, the mass displaces positively, and the total restoring acceleration is negative and acts to return the system to equilibrium. The total restoring accelerations that were identified with the proposed method agree exceptionally well with those accelerations calculated with the modified restoring force method, so this gives confidence to the model for individual restoring accelerations based on stiffness terms from $A(t)$. The damping terms were very small, so their contribution to the total accelerations is small, and some method would be needed to verify the model for damping for this system.

![Total Restoring Acceleration](image1)

![Periodic Response](image2)

*Figure 11: Time periodic restoring forces acting on the mass in (a) and one cycle of the periodic displacement in (b) for forcing frequency $\omega_T=1.0359$."

Now that the estimated models have been verified, the individual restoring accelerations can be interrogated. The $g_{ij}^x$ acceleration-displacement curve is plotted in Fig. 12, and the inset plot shows a detailed view of the force-displacement curve at the extreme end of its range. In the
figure, the open circles are the values obtained by the NL-LTP method, and the lines are the analytical results. The restoring acceleration is negative with positive displacements. Visual inspection suggests that the curve is predominantly linear, although the prior results have shown that the effective stiffness changes by 50%. The $g_{1,2}^x$ acceleration-velocity curve is plotted in Fig. 13 and has the same format as the acceleration-displacement plot. The analytical result appears purely linear, while the NL-LTP result has some variation about the analytical curve.

![Acceleration-Displacement Curve](image)

Figure 12: Acceleration-displacement curve for the Duffing oscillator system ($\omega_r = 1.0359$ rad/s and $A_{ext} = 0.05$): NL-LTP method (circles) and the analytical acceleration-displacement (line).
Figure 13: Acceleration-velocity curve for the Duffing oscillator system ($\omega_f = 1.0359$ rad/s and $A_{ext} = 0.05$): NL-LTP method (circles) and the analytical acceleration-velocity (line).

4.3.1 Discussion

The Duffing oscillator was excited at a single frequency, $\omega_f = 1.0359$ rad/s, yet in Figs. (5) and (6) one can see that it responds at integer multiples of this frequency as well, so the nonlinearity is certainly active. That periodic response is subtracted to obtain $\ddot{x}(t)$, which we presume to be well approximated as linear time periodic. The response of the system before lifting, shown in Fig. 6, contains several peaks. The largest peak is at 1.1 rad/s, so it can be attributed to the time periodic mode and one would take $|\lambda_1| \approx 1.1$ rad/s. If the system were linear then its linearization about the limit cycle would not depend on time and only this peak would occur. However, this system is nonlinear and hence its linearization is time periodic giving rise to several other peaks in the response corresponding to $\lambda_1 + im\omega_f$ as explained by Eq. (24). The peak at 0.98 rad/s just to the left of the zero harmonic is the negative second harmonic term which would appear at $1.1 \text{ rad/s} - 2 \times 1.04 \text{ rad/s} = -0.98 \text{ rad/s}$, but it folds back to $+0.98 \text{ rad/s}$. The peak at 3.18 rad/s is the positive second harmonic of the mode at $1.1 \text{ rad/s} + 2 \times 1.04 \text{ rad/s} =$
3.18 rad/s, and so on. One should also note that if the system was purely linear, one would expect the zero harmonic frequency content to occur at the linear natural frequency of 1 rad/s instead of at 1.1 rad/s, but for a time periodic system the response peaks at the Floquet exponent, which is usually not equal to the natural frequency of the linearized system. All of this information could be used to construct a time periodic model for this system’s mode using the information in Fig. 6, but as was mentioned in Section 3.2.4, it is much more convenient to first lift the measurements.

The spectra that result after applying the lifting technique are much simpler to interpret, In particular, the composite spectrum contains only one strong peak and AMI fits a single mode to the peak with a natural frequency of $|\lambda| \approx 0.0624$ rad/s. This is an alias of the dominant peak at 1.1 rad/s in Figure 6 as expected because the lifting technique resamples the measurement, changing the effective bandwidth of the signals. The aliasing is of little consequence because the aliased Floquet exponent provides a perfectly valid description of the linear time periodic system. The model identified by AMI was then used to reconstruct the state matrix and eventually the acceleration-displacement relationship of this nonlinear system. The acceleration-velocity relationship was also estimated, but contains more errors since the damping model in the state matrix was difficult to estimate. For this system, the linear damping parameter is significantly smaller than the stiffness, and the estimation of $A(t)$ damping terms from the state transition matrix seems to be very sensitive to inaccuracies in the identified linear time periodic modes. Despite this, it is significant to note that the nonlinear parameter estimation was performed without having to assume the functional form of the restoring acceleration a priori. The only assumption was that the periodic orbit was stable and that the response could be approximated as linear time periodic about that orbit. One can fit a polynomial to this curve to
estimate terms like $k$ and $k_3$, or this curve could be used directly to simulate the response of the nonlinear system. For example, a cubic polynomial was fit to the curve, resulting in the function

$$g_{1,1}^x(\ddot{x}_d) = -0.245 \ddot{x}_d - 1.400e^{-3} \dot{x}_d^3 - 1.000 \dot{x}_d + 3.000 \times 10^{-4},$$

which has only 1.9% error on the cubic term as well as a small spurious quadratic and constant terms.

### 4.3.2 Remarks on the Effects of Other Excitation Configurations

There are an infinity of other forcing amplitudes and frequencies that could be used to excite the system. It is important to understand which amplitudes and frequencies are the best, how the periodic orbits will change, or whether there are some configurations one should try to avoid. For example, the response of the Duffing oscillator was simulated again, this time using a forcing frequency of $\omega_r = 1.112$ rad/s and initial conditions $[\ddot{x}_d, \dot{x}_d, x] = [-0.4475, -0.9996]^T$, while all other system parameters remained the same. As before, the first integration was used to find the periodic response of the system and no disturbance forces were applied ($A_d = 0$). Figure 14 shows the time response of the simulation. Initially, the displacement response appears to be periodic, but near 200 seconds the response oscillates at a larger magnitude. After this change in the response, there appears to be a transient type oscillation of the system which then settles to a new periodic motion. In the late time history, the system oscillates at higher amplitude than initially.
Figure 14: Response of the nonlinear DO due to harmonic forcing at $\omega_f = 1.1121$ rad/s. The simulated response was initiated from $[\overline{x}_d, \overline{x}_v] = [-0.4475, -0.9996]^T$ at $t=0$ seconds. The system jumps from a lower amplitude orbit to a higher amplitude orbit.

One might attribute the response to simple transient behavior before the system settles to a steady periodic orbit. The fact that the system showed no transient behavior in the early time response and that no additional inputs were added to the system provides evidence that the phenomenon occurring here is due to the nonlinearity in the system. For this forcing configuration, there are two periodic orbits, and it seems that one of them is unstable. The nonlinear parameters could not be identified using the initial periodic orbit from the figure because it is unstable. For each system that one desires to identify with the NL-LTP method, it is important to understand when situations like an unstable periodic orbit will occur. These types of phenomena occur because of bifurcations in nonlinear systems, and bifurcations are intrinsically linked to changes in system parameters.
4.4 Summary

The NL-LTP method was employed to identify the nonlinear parameters of a Duffing oscillator for a single forcing configuration. The procedure was straightforward to apply, especially since many of the steps are directly related to steps in linear time invariant system identification. A model for the time periodic stiffness was successfully estimated and used to estimate a model for the nonlinear acceleration-displacement relationship. The estimated damping contained more errors, which seem to be due to errors in the time integration routine and inaccuracies in the modal parameters that were identified from the estimated time periodic response. However, the procedure conveniently identifies separate models for the stiffness and damping, so the damping could be estimated using some other more trustworthy method. The modified restoring force method verified the model for nonlinear acceleration-displacement relationships, and all the results except those related to damping compared very well to the analytical results.

A preliminary result was also shown for a response configuration of the system where the proposed techniques would not work. The system was driven in an unstable periodic orbit, which eventually jumped to a nearby stable orbit without any disturbance forces. There are also many other possible inputs that one might have applied, some of which may have worked well for the identification and others which may not. The next section explores this, which will help to understand what amplitude and frequency of the input are needed to apply the identification successfully.
5 Selecting the Optimal Forcing Configuration

5.1 Introduction

In the previous section, the Duffing oscillator was measured with input using two different forcing frequencies and there was a significant difference in the measured responses. The difference occurred because of a bifurcation in the system dynamics. A bifurcation is a local or global change in the qualitative structure of a solution of a vector field or map that occurs when a parameter of the system is changed to a critical value, the *bifurcation value* {Guckenheimer, 1983 #63; Hale, 1991 #127. Because of the Duffing oscillator bifurcations, changes in the forcing configuration may drastically change the periodic orbit character, and it is important to understand how this affects the NL-LTP identification. The goal of this section is to address these ideas. In particular, Section 5.2 discusses an analytical bifurcation analysis of the Duffing oscillator. Then, Section 5.3 presents a numerical bifurcation analysis, which is the preferred method for this dissertation, and the numerical method is used to calculate the Duffing oscillator frequency responses and catalog the possible periodic orbits. In Section 5.4, the frequency responses are used to select several key periodic orbits with which to apply the identification. The effect of the disturbance force amplitude and duration are also considered. The results of this section are used to form a set of guidelines for selecting the optimal forcing configuration with which to apply the NL-LTP method.

5.2 Analytical Study of the Duffing Oscillator

Consider the state space differential equation of motion for the Duffing oscillator, which is repeated here without the disturbance force.
The parameter space for this system is in $\mathbb{R}^5$, meaning that five parameters are needed to uniquely define the harmonically forced system (note that time can always be reduced out of the equations). In experimental engineering work, one often seeks to determine the dynamic system parameters, and therefore may only have control over the harmonic forcing. This reduces the parameter space to $\mathbb{R}^2$. The first step is to determine the equilibria of the nonlinear system both with and without the harmonic forcing.

**5.2.1.1 Case 1: $A_{ext} = 0$**

Let the function $f$ be defined as the right hand side of Eq. (63). When the forcing is turned off, the equilibria can be calculated from the following equation.

$$ f(x_d, x_v) = \begin{cases} x_v \\ -2\zeta\omega x_v - \omega^2 x_d - \omega^2 x_v^3 + A_{ext} \sin(\omega t) \end{cases} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (64) $$

This equation is satisfied when $x_v = 0$ and $-x_d(\omega^2 + \omega^2 x_v^2) = 0$. For the Duffing system of this dissertation, all the parameters in Eq. (64) are positive, and this is common for structural dynamics systems (i.e. no negative damping or stiffness effects). Then, the only $x_d$ that satisfies the previous equation is at the origin. Therefore, there is an equilibrium at $\bar{x}^T = [0, 0]^T$ with a linearization

$$ \frac{\partial f}{\partial x} \bigg|_{\bar{x}} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -2\zeta\omega \end{bmatrix}. \quad (65) $$
and eigenvalues $\lambda_{1,2} = -\zeta \omega \pm \omega \sqrt{\zeta^2 - 1}$. The eigenvalues have negative real parts and non-zero imaginary parts, since $\zeta$ and $\omega$ are positive. So, the damped model of this system will have solutions in state space that tend to the origin as time tends to positive infinity. This can be seen in Fig. 15(a). Two solution curves were calculated for the full nonlinear system by numerically integrating the equations of motion forward in time. The positive flow of time is depicted with arrows on the solution curves. The undamped model can easily be determined from the results for the damped model by setting $\zeta = 0$. Without damping, the system will be nonhyperbolic {Guckenheimer, 1983 #63} since its linearization has purely imaginary eigenvalues $\lambda_{1,2} = \pm i \omega$. This system is purely oscillatory with solutions flowing periodically about the origin, and two state space solution curves are depicted in Fig. 15(b).

![State Space Solutions Curves with Damping](image1)

![State Space Solutions Curves without Damping](image2)

Figure 15: Plot of two solutions curves in state space of the damped (a) and undamped (b) nonlinear system in Eq. (63)($A_{ext}=0$). The direction of increasing time is shown with arrows.

Considering the physics of the problem, these are the only possible response equilibrium configurations of the system for the unforced case. A change in magnitude of any of the parameters will not change that the system is oscillatory stable with one equilibrium when damping is present and purely oscillatory without damping.
5.2.1.2 Case 2: $A_{ext} \neq 0$

Suppose that the harmonic forcing is turned on with magnitude $A_{ext}$ and frequency $\omega_f$. The right hand side equation is the following.

$$f(x_d, x_v) = \left\{ \begin{array}{l} x_v \\ -2\zeta\omega x_v - x_d (\omega^2 + \omega_f^2 x_d^2) + A_{ext}\sin(\omega_f t) \end{array} \right\} = \{0\}.$$  \hspace{1cm} (66)

The two conditions for satisfying this equation are $x_v = 0$ and the following equation

$$-x_d (\omega^2 + \omega_f^2 x_d^2) + A_{ext}\sin(\omega_f t) = 0.$$  \hspace{1cm} (67)

The previous condition is satisfied for $(x_v = 0, A_{ext} = 0)$ or $(\omega = \omega_f = 0, A_{ext} = 0)$, so there are no equilibria for $A_{ext} \neq 0$. However, the harmonic forcing will cause periodic orbits as discussed in Section 2.3.1, and the system could bifurcate as the forcing amplitude and frequency change and combine in different configurations. For example, Fig. 14 showed an example in which the system had two periodic orbits for the particular $(A_{ext}, \omega_f)$ configuration. In order to find the bifurcation values of forcing amplitude or frequency, one must be able to compute the periodic orbits for many combinations of forcing frequency and amplitude. One way to compute the periodic response of Eq. (63) is to assume an series solution. For example, one widely used analytical technique is called the Harmonic Balance Method [17, 63]. First, one assumes a solution with the form

$$x_d(t) = \sum_{m=0}^{N_h} a_m \cos(m\omega_f + m\beta_0)$$  \hspace{1cm} (68)

where $a_m$ is an amplitude coefficient and $\beta_0$ is a phase term. The assumed solution is differentiated to obtain $x_v(t)$ and $x_a(t)$, and all series are inserted into the lower half of Eq. (63).
Then, one then assumes all coefficients $a_m = 0$ for $m$ greater than $N_h$, equates like-terms on the left and right hand side of Eq. (63) in order to achieve a system of equations, and solves for the unknown coefficients $a_m$ and phase $\beta_0$. It could be quite time consuming to use this method to calculate a whole range of periodic solutions for different frequencies and amplitudes of the input. Additionally, one must exercise a certain amount of expertise in order to correctly truncate the assumed series for the harmonic solution.

5.2.2 Discussion

Even the simplest dynamical systems can display complex changes in dynamics when parameters are varied. The methods presented above provide a valuable approach for analytical bifurcation analysis. In order provide more details on this type of approach, Appendix 1 contains analytical bifurcation analyses of two systems, one that produces the saddle node bifurcation and one that produces the pitchfork bifurcation. These examples are somewhat academic, but they provide some fundamental techniques that can be used for more complicated systems. A number of researchers have used these methods to study real systems, for example in [64-66], so the methods are certainly worth mentioning.

The Duffing oscillator is perhaps the simplest nonlinear structural system one can imagine, and yet there is no closed form representation of its bifurcation analysis. In order find these bifurcations, explore the periodic dynamics for different forcing configurations, and determine the optimal forcing configuration for the identification, a numerical continuation technique was developed and is discussed below.
5.3 Numerical Continuation to Calculate Periodic Orbits

In order to analyze a variety of input amplitudes and frequencies, a numerical continuation technique is used to compute all the periodic orbits of the Duffing oscillator over a range of forcing frequencies (i.e. a frequency response). This method is presented in detail in Appendix 2 (a discussion of analytical methods and alternative numerical methods is also provided). Each state vector of initial conditions at a given forcing frequency on the frequency response curve defines a periodic orbit. The system can be integrated in time from the initial conditions while being forced at the corresponding forcing frequency, and a periodic orbit will be produced.

The basic procedure to sequentially calculate a whole branch of periodic solutions is the following. First, the user supplies an initial guess (e.g. a low frequency solution from a static analysis), and a Newton-Raphson correction technique updates the guess to find the first state vector values on the solution branch. Then, all subsequent solutions in a frequency range are computed with an iterative procedure: 1) the procedure first calculates a prediction for the next state vector and forcing frequency pair on the periodic solution curve; 2) the procedure calculates Newton-Raphson correction steps to update the prediction values until the state vector and forcing frequency pair resides on the periodic orbit solution curve; and the procedure repeats the first two steps for the next periodic orbit. The process continues until the desired frequency band has been characterized. This procedure is based on commonly used numerical methods [67], and is structured after recently developed method for calculating the periodic orbits of unforced nonlinear systems [33]. The continuation procedure efficiently calculates a full frequency response curve that provides not only the forcing configuration bifurcation values but also all the periodic orbit conditions for an entire forcing frequency band.
5.3.1 Continuation of Duffing Oscillator Periodic Orbits

The continuation algorithm was used to calculate the frequency response of the Duffing system described by Eq. (63) for a range of forcing frequencies between $\omega_f = 0.1$ and 10 rad/s. The parameters used for the oscillator were $\omega_1 = 1$, $\omega_2 = 0.5$, and $\zeta = 0.01$, and the forcing amplitude was $A_{ext} = 0.05$. The starting guess for the algorithm was calculated by computing the forced response of the underlying linear system with $\omega_f = 0.1$ rad/s. The algorithm was then allowed to run in an automated fashion until the forcing frequency exceeded 10 rad/s. Since the linear natural frequency of this system is $\omega = 1$ rad/s, the largest responses are expected to occur within this frequency band.

Figure 16 shows the initial conditions $[\bar{x}_d, \bar{v}_r]^T$ that produced a periodic response at each forcing frequency. All the displacement initial conditions are plotted as one curve versus forcing frequency (blue line), and the velocity initial conditions are plotted as a separate curve (green line). The forcing frequencies in all of the following plots are normalized by dividing by the linear natural frequency $\omega = 1$ rad/s so that the abscissa is given by the dimensionless quantity $\omega_f / \omega$. Each displacement and velocity pair could be integrated over $0 \leq t \leq T$ to find the complete periodic orbit. As described in Appendix 2, the stability of each periodic orbit is also calculated by the continuation procedure, and the stable periodic orbits are shown with solid lines while the unstable periodic orbit solutions are shown with dashed lines. The arrows in the plot designate specific frequency lines, which are labeled by frequency and will be discussed later.

The distinct resonance peak in the displacement curve is similar to that of a linear, single degree of freedom frequency response function, but bends slightly to higher frequencies (i.e. the system is stiffening [17]). Moreover, the solutions on the lower frequency arm of the peak are
stable while some of the solutions on the higher frequency arm of the peak are unstable. When the displacement peaks near $\omega_r / \omega = 1$, the velocity magnitude becomes large and negative, but as the frequency increases the velocity goes through a zero and then sharply becomes large and positive.

![Figure 16: Initial conditions that result in periodic responses for $A_{ext}=0.05$ plotted versus dimensionless forcing frequency.](image)

The solutions curves in Fig. 16 are multi-valued functions of the frequency, $\omega_r$, in the band where the resonance peak bends to higher frequencies. A vertical slice through the curves gives the appropriate initial state vector values to produce a periodic response at the particular frequency. In the multi-valued region, there are up to three periodic orbits per forcing frequency, so the system clearly bifurcates as the forcing frequency is varied.

### 5.3.1.1 Tracking the DO Periodic Orbits

The periodic orbits at different values of $\omega_r$ were studied in order to analyze the bifurcations in the previous frequency response. Specifically, at each of the frequencies labeled in Fig. 16 (i.e. $\omega_r / \omega = 0.65, 1.0, 1.07, 1.15, 1.3, \text{and } 1.4$), all corresponding state vector pairs
were used to compute one cycle of periodic response for that frequency. The periodic orbits for each frequency are plotted in Fig. 17(a-f) in the phase plane. The stable orbits are plotted with a solid line and the unstable orbits with a dashed line.
Figure 17: Periodic orbits of the Duffing oscillator in state space for different driving frequencies $\omega_T$. Subplots (a-f) each correspond to periodic solutions for the labeled frequencies in Fig. 16.
The different subplots show how the periodic orbits change as the force frequency is increased. When $\omega_r/\omega = 0.65$, the system has a single, small amplitude periodic orbit (Fig. 17(a)). When the frequency was increased to $\omega_r/\omega = 0.65$, the period orbit increased in size (Fig. 17(b)) but looks qualitatively the same. In Fig. 17(c) for $\omega_r/\omega = 1.07$, two periodic orbits are displayed. This frequency is an approximate bifurcation value. A vertical line section (shown with a light gray line) at $\omega_r/\omega = 1.07$ intersects the lower frequency branch of the displacement resonant peak and is approximately tangent to the higher frequency branch of the peak (shown with the red dots). Similar observations can be made for the velocity curve. Theoretically, the periodic orbit at the bifurcation location is marginally stable (Floquet exponent with zero real part). This frequency corresponds to the birth of a cyclic fold bifurcation such that an additional periodic orbit is created for frequencies just higher than the bifurcation value. In particular, three orbits are present at $\omega_r/\omega = 1.17$ in Fig. 17(d); there are two stable orbits (outside and inside orbit) and one unstable orbit (intermediate orbit). The unstable orbit is nearly the same size as the outer stable orbit. It has increased in size since its creation at $\omega_r/\omega = 1.07$. As the frequency gets closer to the resonance condition, both the outer two orbits increase in size and then coalesce at $\omega_r/\omega = 1.3$. Here, two periodic orbits are destroyed. The unstable orbit merges with the outer stable orbit to create the large, marginally stable periodic orbit in Fig. 17(e). For frequencies higher than $\omega_r/\omega = 1.3$, there is only one periodic solution. The only remaining orbit for high frequencies has very small magnitude in phase space. This orbit is shown at $\omega_r/\omega = 1.4$ in Fig. 17(f).
5.3.1.2 Discussion

The creation and destruction of periodic orbits in the Duffing frequency response has an important effect on the operating conditions of the system. The multi-valued region in the frequency response contains the states for unstable periodic orbits. So, one expects the identification method to fail if it is performed using any of those unstable orbits. In fact, this was exactly the case that occurred for the jump response shown by Fig.14. There are still many other stable periodic orbits on the frequency response that could be used for the identification. However, not all of these produce large displacements that activate the nonlinearity, and one may desire to identify the nonlinearity over a larger displacement than is activated by the largest amplitude orbit on the given frequency response. So, in order to also evaluate how the forcing amplitude affects the bifurcation dynamics, additional nonlinear frequency responses can be calculated.

5.3.2 Duffing Oscillator Frequency Responses

The continuation algorithm was used to calculate additional frequency response curves for the Duffing system using forcing amplitudes of $A_{ext}$=0.01, 0.1, and 1. For each amplitude, the frequency response curve was calculated between $\omega_r/\omega$=0.1 and 10. The displacement magnitude frequency response curve for each forcing amplitude is plotted in Fig. 18, where the curves for $A_{ext}$= 0.01, 0.1, and 1 are shown with blue, green, and red lines, respectively. The backbone curve for the unforced and undamped system [17, 33] is shown with a black line and it crosses each frequency response at the maximum resonance value. The red curve ($A_{ext}$= 1) has a few additional peaks below $\omega_r/\omega$=0.5, and the inset plot contains a detail image of those peaks. These peaks are superharmonic resonances [17, 68], and they occur when the forcing amplitude
is high and the forcing frequency is near an integer fraction of the linear natural frequency of the system. For example, the periodic orbit for $A_{ext}=1$ and $\omega_r=0.393$, which is labeled in Fig. 18 with the black dot) is plotted in the phase plane in Fig. 19, and this phase portrait reveals that the orbit is significantly more complicated than any of the previous orbits that were dominated by a single frequency. The superharmonic orbit has large amplitude in the displacement and there are three loops in the phase portrait because the drive frequency harmonic $3\omega_r$ also excites the dominant resonance of the system.

![Figure 18: Nonlinear normal mode backbone curve and nonlinear frequency responses of the Duffing oscillator for forcing amplitudes $A_{ext}=0.01$, 0.1 and 1.](image-url)
5.3.2.1 Discussion

As the forcing amplitude was varied, most of the frequency response curves were qualitatively similar to the curve for forcing amplitude $A_{ext}=0.05$, although there were a few important differences. As the forcing magnitude was increase, the maximum magnitude of the resonance peak increased, but also the multi-valued region of the frequency response curve increased since the system is stiffening. Additionally, the range of the unstable branch of solutions increased with the forcing amplitude. Finally, the superharmonic resonance peaks designate a forcing configuration where relative large amplitude response is possible at a forcing frequency away from the linear natural frequency. All of these nonlinear dynamics can be visualized with the frequency response curves. These curves will be useful to explore the periodic orbits of different forcing configurations to see how the forcing configuration effects the nonlinear identification.
5.4 Nonlinear system Identification Based on Different Periodic Orbits

5.4.1 Effect of Different Excitation Configurations

In the previous section, several frequency response curves were computed for the Duffing system, and it was shown that the periodic orbits and response of a nonlinear system depend strongly on several factors including the drive force amplitude and frequency. But, the method in this work also requires a disturbance force with an amplitude and duration as well. It is therefore necessary to consider how the amplitude and duration of the disturbance affect the nonlinear identification proposed here. So, the frequency responses will be used to select specific forcing configurations in order to explore the nonlinear identification. Figure 20 shows the frequency response curve of the Duffing oscillator for amplitude $A_{ext}=0.1$. The figure clearly shows a superharmonic resonance peak and the spring hardening region (shaded region), which is also the multi-valued region of the curve. In Section 4.3, it was shown that the identification could be applied successfully when the system was excited in a region just below resonance (with harmonic forcing amplitude $A_{ext}=0.05$), and the corresponding region for $A_{ext}=0.1$ is labeled below. In Sections 5.4.2-5.4.4, the identification will be explored for two orbits in the multi-valued region corresponding to the same forcing frequency, MS Orbit 1 and MS Orbit 2 in Fig. 20, for the Superharmonic orbit shown in Fig. 19 that is similar to orbits on the superharmonic resonance peak labeled in the figure below, and for excitations that exhibit period lengthening effects (such as period doubling, [34] pages 70-91).
Figure 20: Spring hardening nonlinear frequency response with superharmonic resonance for harmonic forcing amplitude $A_{\text{ext}}=0.1$.

5.4.2 Excitation Producing Multiple Well-Separated Orbits

As mentioned previously, this system has different types of periodic solutions depending on the forcing frequency. In the shaded region of Fig. 20 (except at the boundaries), there are three possible responses for a single forcing frequency, one of which is unstable. It was already shown in Section 4.3.2, that an unstable periodic orbit is not suitable for identification by this method because the response of the system tends away from the unstable orbit towards a nearby stable orbit, even without a perturbation. When multiple stable periodic orbits are possible for a single forcing frequency, the identification can theoretically be performed using any of the periodic orbits, presuming that the perturbations are small enough to keep the response of the system close to the originating orbit.

This was explored using a drive frequency and amplitude of $\omega_f=1.3566$ rad/s and $A_{\text{ext}}=0.1$, which is a configuration that has stable periodic orbits that can be initiated from $[\bar{X}_d, \bar{X}_r]^T = [-0.0039, -0.1622]^T$ (Multiple Solutions orbit 1) and $[\bar{X}_d, \bar{X}_r]^T = [-1.2019, 2.402]^T$. 

\[\begin{array}{l}
\end{array}\]
(Multiple Solutions Orbit 2). Figure 21 shows the state space portraits of the two stable periodic orbits for this forcing configuration. The response in Multiple Solution Orbit 2, shown with the dashed curve (MS Orbit 2), has much greater amplitude than that of Multiple Solution Orbit 1 shown in with the solid curve (MS Orbit 1).

![Phase Space Portraits of the Periodic Orbits](image)

*Figure 21: State space portraits of two stable periodic orbits that exist for a forcing frequency $\omega_r = 1.3566$ rad/s at forcing amplitude $A_{ext}=0.1$."

Following the exact same procedure as in Section 4.3, the state coefficient matrices were identified for both these periodic orbits. The stiffness terms are shown in Fig. 22 with the values for the MS Orbit 1 plotted with filled squares, the values for MS Orbit 2 plotted with open circles, and the values from the analytical model plotted with solid lines. The acceleration-displacement relationship was constructed for each identified model and those curves are shown in Fig. 23. Since the modified restoring force method was explained with Identification Case 1, only the analytical results will be used to verify the remaining cases with the Duffing system. If the estimated results agree with the analytical results, it is assumed that the results would also be verified with the modified restoring force method. The identified system for the MS Orbit 2 is significantly time periodic, since the stiffness varies over the period by 450% of the initial
stiffness value, while the system for MS Orbit 1 is approximately linear time invariant. As a result, the acceleration-displacement curve for MS Orbit 2 is very nonlinear, while the force displacement curve for MS Orbit 1 is very linear and only captures the force-displacement relationship over a small range. Both of the identified models compare very well to their corresponding analytical state coefficient models, verifying that the proposed identification method works even when the input produces multiple periodic orbits that are stable and well separated.

Figure 22: Components of the state coefficient matrix that correspond to the instantaneous stiffness of the system for an entire periodic orbit for MS Orbit 1 and 2 (\(\omega_T=1.3566\) rad/s and \(A_{ext}=0.1\)).
5.4.3 Superharmonic Resonance

Next, the identification will be applied to a periodic orbit from a superharmonic resonance, where an excitation at some fraction of the linear natural frequency excites a resonant response. This was explored for the Duffing oscillator using the orbit in Fig. 19 which corresponds to an amplitude of $A_{ext}=1$ and a driving frequency of $\omega_f=0.3934$ rad/s. The initial conditions $[\bar{x}_d, \bar{x}_v]^T = [-0.5695, 1.1170]^T$ were used as the starting point when computing both the periodic and perturbed responses. The same impulsive forcing conditions were used. The solutions were evaluated with a sampling frequency of approximately $f_{samp}=2.76$ Hz, which results in 44 samples per cycle of the harmonic response. Both responses were evaluated for a time window length containing 614 full cycles of the harmonic response frequency. Figure 24 shows the time responses of the Superharmonic orbit in the same format as Fig. 4. The time histories show that the resulting periodic and perturbed responses contain at least two prominent frequencies.
Figure 24: Response of the nonlinear Duffing oscillator excited by harmonic ($\omega_r = 0.3934$ rad/s and $A_{\text{ext}} = 1$) excitation near a superharmonic resonance and impulsive excitation. Plot (a) provides the initial cycle of the perturbed response (red line) and the periodic response (blue with dots). Plot (c) is the approximate linear time periodic response found by subtracting the two signals in (a).

The lifting technique was applied and the 44 lifted responses were processed with AMI. AMI identified an eigenvalue of $\lambda_1 = -0.0079 + 0.0343i$, a natural frequency of $|\lambda_1| = 0.0352$ rad/s, and a damping ratio of $\text{Re}(-\lambda_1)/|\lambda_1| = 0.2243$ for the linear time periodic system. The Fourier series expansion method was then applied to the extracted mode. Figure 25(a) shows amplitudes of each of the coefficients in the Fourier series expansion of the mode (open blue circles). Several Fourier terms have large amplitudes, but there are some Fourier terms with smaller amplitudes that seem to be distinct from the noise floor. Only the meaningful Fourier terms should be retained to construct $\Phi(t, t_0)$ and $A(t)$. Figure 25(b) shows the spectra of the time periodic response from the Superharmonic simulation. The identified time periodic mode was attributed to an aliased version of the peak at 1.21 rad/s, and this is therefore the $m=0$ term in the Fourier series expansion. This peak and a few others are labeled with their frequency and
Fourier coefficient number, $m$. Using this information, several of the Fourier expansion terms can be selected to retain based on the corresponding coherent peak in the spectrum (e.g. the $m=+2$ and $+4$ terms). A few other peaks such as that at 0.42 rad/s and 4.3 rad/s have smaller magnitude and are less coherent. The correlation metric described in Section 3.2.1 for the Fourier series expansion method was used to discern if the Fourier terms for these peaks should be retained. The subtraction residual of the peak in the measurement [22] was correlated with a linear time invariant single mode fit using Eq. (26), and the Fourier term was retained if the peak correlation value was at least 0.8. For example, the peak at 0.42 rad/s ($m=-2$) had a correlation value of 0.93, so it was retained, but the peak at 4.3 rad/s ($m=-14$ since 1.21-14*0.3933=-4.296) had a correlation value of 0.746, and it was discarded. Based on the peaks in the spectrum and the correlation results, the red dots in Fig. 25(a) are the coefficients of the expansion terms that were retained when computing $A(t)$. 
Figure 25: (a) Fourier Series expansion of the linear time periodic model from the Superharmonic orbit ($\omega_c = 0.3934$ rad/s and $A_{ext} = 1$). (open blue circles) Fourier coefficients of the identified mode. (solid red circles) Dominant Fourier coefficients that were retained when creating $A(t)$ and $\Phi(t, t_0)$. (b) Spectrum of approximated time periodic response. A few peaks are labeled with frequency and corresponding Fourier coefficient number.

Figure 25 shows that 10 Fourier terms are used to describe the identified mode shape. The state transition and state coefficient matrix were constructed using these 10 terms. Figure 26(a) shows the stiffness and damping terms in the state coefficient matrix that were identified by the proposed method, and 26(b) shows one cycle of the periodic displacement for reference. The identified damping varies about the constant analytical value, 0.02, by 500% of 0.02 at certain instants and also changes signs at certain instants, although it seems to agree with 0.02 in an average sense. The stiffness of the system is highly time periodic, varying by up to 250% of
its initial value. It also varies in a more complicated way than it did in Fig. 9, since the periodic orbit is more complicated; each time the system displaces to a local extremum the stiffness increases to a local maximum. The stiffness coefficients identified with the proposed frequency domain technique compare very well to the analytical results.

Figure 26: Components of the state coefficient matrix that correspond to the instantaneous stiffness (component (2,1)) and damping (component (2,2)) of the system for the Superharmonic orbit ($\omega_f = 0.3934$ rad/s and $A_{ext} = 1$).

Figure 27 shows the identified stiffness coefficient plotted versus the displacement of the system (open circles), and Fig. 28 shows the acceleration-displacement curve that was the result from integrating the state coefficient matrix to reconstruct the nonlinear parameter functions. The values for the identification from the Superharmonic orbit are shown with open red squares, the corresponding values from all other identification cases are shown (Identification Case 1-
open black diamonds, MS Orbit 1-filled blue squares, and MS Orbit 2-open blue circle, and the analytical results based on the Superharmonic orbit are also shown. These results show that although the stiffness of the system varies with time in a much more complicated way, the stiffness versus displacement relationship still seems to follow a quadratic relationship and the restoring acceleration relationship seems to be cubic as before. It is noteworthy that one equation of motion can produce such complicated and drastically different periodic orbits, and furthermore that the method is able to identify a result that is so similar to that obtained from the much simpler orbits that were studied in Sections 4.3 and 5.4.2.

![A(2,1) Component](image)

*Figure 27: Displacement varying terms from the estimated (circles) and analytical $A(t)$ (line) matrix plotted versus the displacement for the Superharmonic orbit ($\omega_f=0.3934$ rad/s and $A_{ext}=1$).*
5.4.4 Effect of Period Doubling, Tripling, etc...

Several texts on nonlinear systems theory mention the fact that a system with a stable periodic orbit can sometimes respond with a period that is longer than that of the input. Typically, as some parameter such as the forcing amplitude is increased, one observes that the fundamental frequency of the response, which is initially \( \omega_r \), first becomes \( \omega_r / 2 \), then \( \omega_r / 3 \), etc... so that the fundamental period of the orbit doubles, then triples, etc... [34, 69]. Unfortunately, this type of behavior is difficult to predict analytically, but it certainly deserves consideration since it might affect the proposed system identification strategy.
The authors have observed period lengthening when simulating identification of the Duffing system. For example, this was observed in Multiple Solutions Orbit 2 from Section 5.4.2. Figure 29 shows the frequency spectrum of the simulated periodic plus perturbation response for that case. As was noted for the spectrum shown in Fig. 6, the response spectrum has sharp peaks at the input frequency, $\omega_r=1.356$ rad/s and its third harmonic, $3\omega_r=4.068$ rad/s, and also shows the presence of the linear time periodic mode at 1.27, 1.44, 3.98, and 4.15 rad/s. However, this spectrum also shows several other peaks. The lowest appears at 0.452 rad/s and all of the others are integer multiples of this frequency: 2.260, 3.164, and 4.972 rad/s. Relative to the drive frequency, 0.452 rad/s is $\omega_r/3$ and the other frequencies are $(5/3)\omega_r$, $(7/3)\omega_r$ and $(11/3)\omega_r$. This spectral content could only be present if the response of the system has a period that is three times as long as the period of the forcing, revealing that this is a case of period-tripling. In {Guckenheimer, 1983 #63}, Guckenheimer and Holmes show that the Duffing system can produce the strange attractor, and that stable period 3T orbits are also possible for
system parameter values that are near those that produce the strange attractor. This is a possible explanation of the orbit which appears to be period $3T$ in the plot above. One could account for this in the proposed identification scheme by simply noting that the period of $\bar{x}(t)$ is actually three times longer than the period of the input. All other steps in the proposed identification algorithm could remain the same. However, it is also possible the new length $3T$ periodic orbit remains very close to the original length $T$ orbit even though the period has lengthened and if this is the case then one might be able to ignore the fact that the period has lengthened.

To investigate this, the phase portrait of the orbit, which was simulated in the time domain, is plotted in Fig. 30 over many cycles of the response. Surprisingly, this appears to be the same as the original length $T$ limit cycle. Closer inspection reveals that there are actually three curves, as illustrated by the two inlaid plots. These curves could be traced around the phase plane, revealing that one must complete three loops before the response repeats itself. However, these curves always remain very close to the original length $T$ orbit. Since the state of the system is practically the same at each point along the length $T$ and length $3T$ orbits, one can ignore the period lengthening in this case and process the measurement as described in the previous section. However, if one takes this approach, then the $\omega_r/3$ (or, in general, $\omega_r/N$) harmonics in the spectrum would not be removed when subtracting $\bar{x}(t)$ from the measurements. They should be ignored when performing system identification since they are a property of the orbit and not of the time periodic approximation about that orbit. If this is done, good results can still be achieved as was shown in Section 5.4.2.
5.4.5 Effect of Excitation and Perturbation Amplitudes

As explained in Section 2.3.1, the time periodic model that this method is based upon is only valid if the system remains near the periodic orbit. However, the periodic orbit dictates the range of the measurement sensors, so one would like make the perturbation as large as possible to assure that it is not buried in measurement noise. Furthermore, the previous sections have shown that the excitation affects the complexity of the time periodic model, dictating the number of harmonics that must be extracted from a measurement to adequately characterize the system. This section explores these issues for the Duffing system by varying the periodic orbit and the perturbation from that orbit and evaluating the effect on the identified time periodic model.

The strength of the perturbation from the periodic orbit is quantified by computing the ratio of the amplitudes of the response and the perturbation in the frequency domain, which is denoted $\alpha$

$$\alpha = 100 \left( \frac{\max \{\tilde{X}(\omega)\}}{\max \{X(\omega)\}} \right)$$  \hspace{1cm} (69)
where the capital letters $\tilde{X}$ and $X$ denote that these are the Fourier transforms of $\tilde{x}(t)$ and $x(t)$ respectively and multiplication by 100 makes $\alpha$ a percentage. This is illustrated in Fig. 6.

The complexity of the time periodic system model can be characterized by the Fourier coefficients of the time periodic mode shape (e.g. those shown in Figs. 8 and 25). The relative amplitudes give a measure of the complexity so these are denoted $B_n$ for any integer $n$. The simplest possible model for the system about the limit cycle is a time invariant one, where $B_0 = 1$ and $B_n = 0$ for all other $n$. The coefficients for $n \neq 0$ generally become larger as the complexity of the time periodic system model increases, as was illustrated in Sections 5.4.2 and 5.4.3.

These metrics were calculated for forcing amplitudes $A_{ext} = 0.01$ and 0.1 in Eq. (61), forcing frequencies $\omega_f = 0.3802$ and 0.9868 rad/s, and perturbation magnitudes that gave $\alpha = 0.5\%$ and 2\%. Table 1 provides the results. The two driving frequencies considered are near one third the linear natural frequency and just below the linear natural frequency, similar to those that were found to produce good results in Sections 4.3 and 5.4.3. The top block labeled (a) gives the results for driving force amplitude $A_{ext} = 0.01$ while the bottom block corresponds to $A_{ext} = 0.1$. The relative amplitudes of the Fourier coefficients $B_2$, $B_0$ and $B_2$ are shown for each case. These were obtained by applying a curve fit to the lifted spectrum as described in the previous sections and then finding the ratio of the Fourier coefficient amplitudes. The analytical Fourier coefficients were also calculated for each case and are shown.
(a) $A_{\text{ext}}=0.01$

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<thead>
<tr>
<th>$\alpha$ (%)</th>
<th>Analytical</th>
<th>0.5%</th>
<th>2.0%</th>
<th>Analytical</th>
<th>0.5%</th>
<th>2.0%</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_{-2}$</td>
<td>0.0000</td>
<td>0.0001</td>
<td>0.0000</td>
<td>0.1141</td>
<td>0.1124</td>
<td>0.1018</td>
</tr>
<tr>
<td>$B_{0}$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$B_{2}$</td>
<td>0.0000</td>
<td>0.0002</td>
<td>0.0001</td>
<td>0.0014</td>
<td>0.0020</td>
<td>0.0035</td>
</tr>
</tbody>
</table>

(b) $A=0.1$

<table>
<thead>
<tr>
<th>$\omega_\tau$</th>
<th>0.3849 rad/s</th>
<th>0.9821 rad/s</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$ (%)</td>
<td>Analytical</td>
<td>0.5%</td>
</tr>
<tr>
<td>$B_{-2}$</td>
<td>0.0027</td>
<td>0.0027</td>
</tr>
<tr>
<td>$B_{0}$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$B_{2}$</td>
<td>0.0012</td>
<td>0.0010</td>
</tr>
</tbody>
</table>

Table 1: Fourier coefficients of the identified linear time periodic mode for various drive amplitudes, $A_{\text{ext}}$, frequencies, $\omega_\tau$, and perturbation amplitudes, $\alpha$. The results show that the Fourier coefficients begin to differ from the true, analytical values if the perturbation amplitude is too large.

When $A_{\text{ext}}=0.01$ and $\omega_\tau=0.3802$ rad/s the response is almost purely linear and the only significant Fourier term is the linear term ($B_0$). The Fourier coefficients for both low ($\alpha=0.5\%$) and larger ($\alpha=2\%$) deviation from the periodic orbit are less than 0.0002, so the identification procedure would apparently be accurate under these conditions, although not all that useful since the nonlinearity is not well excited. When the excitation frequency is increased to $\omega_\tau=0.9868$ rad/s, the $B_{-2}$ harmonic is approximately 11\% as strong as the dominant term indicating that the nonlinearity is active. When the perturbation amplitude is $\alpha=0.5\%$, the $B_{-2}$ and $B_{2}$ coefficients are identified within 0.002 and 0.0006 of the analytical values, respectively. However, when the perturbation is increased to $\alpha=2\%$, the identified Fourier coefficients differ from the true values by more than 0.01 and 0.002 respectively. Even at this level of $\alpha$, the reconstructed force-displacement curves were still reasonable which shows that the method is quite robust to errors in the identified Fourier coefficients.
Increasing the magnitude of the driving force to $A_{ext}=0.1$ increases the coefficients $B_2$ and $B_2$ for all of the cases, indicating that the nonlinearity is more active and so the time periodic approximation is more complicated. In all of these cases, when the perturbation amplitude is small, $\alpha=0.5\%$, the Fourier coefficients are accurate to within 0.0021 for $B_2$ and 0.0002 for $B_2$. When the perturbation amplitude was increased to $\alpha=2\%$, the errors in the coefficients increase to 0.04 and 0.004 respectively. For this system it seems that the perturbation amplitude must be about 0.5\% or less in order to accurately identify a time periodic model for the system. Hence, fairly accurate sensors will be needed to apply this method in practice. Nevertheless, the entire NL-LTP procedure was performed for each case in Table 1, and in each case, trustworthy nonlinear models were estimated using the procedure. For example, the acceleration-displacement curve estimated from the case with $A_{ext}=0.1$, $\omega_r = 0.9821 \text{ rad/s}$, and $\alpha=2\%$ is shown in the figure below. The estimated result compares very well to the analytical result despite any errors in the identified linear time periodic model.
Summary

The results show that the proposed nonlinear identification methods can be successfully employed to identify nonlinear system parameters if the conditions are appropriate. The following observations were made.

1. Identification was successful under a broad range of excitation frequencies and amplitudes, so long as the amplitude of the periodic orbit was large enough to exercise the nonlinearity.
2. Some periodic orbits are more complicated than others, for example when the system is excited at a superharmonic resonance. This makes the underlying linear time periodic model more complicated as well, perhaps increasing the chance that an important harmonic might be missed, but the cases studied here still seemed to produce good results.
3. One forcing configuration was observed, in a region where multiple responses were possible for a single forcing frequency, where the system’s response had a larger period than the input. Even in this case, the system was accurately identified because the
lengthened orbit remained close to the original orbit.

4. The linear time periodic approximation is accurate if the system’s response remains very close to the periodic orbit. For the Duffing system, adequate results were obtained if the amplitude of the perturbation was less than 0.5% of the periodic orbit amplitude. Hence, fairly accurate sensors will be needed to implement the method experimentally.

The identification extends very naturally to higher order systems. As demonstrated here, the measurements are easy to interpret in the frequency domain, and as with familiar time invariant systems, additional degrees of freedom simply manifest themselves as additional peaks in the spectra. The system identification routines used to extract the time periodic modes can readily accommodate measurements with relatively large numbers of modes. On the other hand, higher order systems can exhibit even more complicated periodic orbits, so one will have even more freedom in selecting the optimal periodic forcing function. These issues are explored in the next sections.

6 Application to a Nonlinear Cantilever Beam

6.1 Introduction

In order to evaluate the NL-LTP identification method for multi-degree-of-freedom systems, it was applied to measurements from a nonlinear beam. Figure 32 below shows a top view photograph of the experimental setup. An aluminum 6061 alloy beam is bolted to a steel mounting block, shown on the left side of the image. A small strip of nylon is bolted to the free end of the cantilever and clamped to another steel mounting block. Both of the mounting blocks are bolted to a massive steel tube. The steel tube and mounting blocks approximate the fixed support of an ideal cantilever. The whole setup rests on a foam pad on a massive table top. The
beam is oriented such that the bending axis is parallel to the plane of the table top. Figure 33 shows a close top and front view of the nylon strip between the tip of the beam and the right hand side support. Table 2 below provides the physical dimensions of the beam and the nylon in millimeters.

Figure 32: Top view of the experimental nonlinear beam setup.

Figure 33: Top view (a) and front view (b) of the nylon strip spring connected to free end of the cantilever beam.

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Al 6061 Beam</th>
<th>Nylon 6/6 Strip</th>
<th>1095 Spring Steel*</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length</td>
<td>1016</td>
<td>53.2</td>
<td>53.2</td>
</tr>
<tr>
<td>Width</td>
<td>25.4</td>
<td>25.4</td>
<td>25.4</td>
</tr>
<tr>
<td>Thickness</td>
<td>9.5</td>
<td>0.254</td>
<td>0.0762</td>
</tr>
</tbody>
</table>

Table 2: Dimensions, in millimeters, of the 6061 Aluminum beam and the nylon strip. *The original experimental setup used blue-finished and polished 1095 spring steel at the free end, but this was later changed to a nylon strip.
The nylon strip spring on the tip of the beam adds stiffness at that point that depends nonlinearly on the tip displacement. This setup was originally proposed in [70], although their beam had a strip of spring steel instead of nylon, and other researchers have studied similar beam setups [6, 20, 71]. Spring steel was also used instead of nylon in the author’s preliminary work [72]. However, it was difficult to control the initial tension in the spring steel, and if it became loose one would see a very sharp increase in stiffness when the spring steel became tight (at a certain deflection of the tip). This critical point acted as a limit, since no applied force could cause the beam to deflect beyond that point. As a result all of the periodic orbits that were obtained were very complicated, and it was difficult to ascertain whether the identification results were correct. For these reasons, the nylon strip will be used instead of the steel in all of the experimental results presented in this dissertation, although the parameters for the steel strip will be retained in the analytical model that is presented in the following section.

Figure 34 below shows a schematic of the system, which is modeled as a uniform, prismatic cantilever beam with material density $\rho$, elastic modulus $E$, cross sectional area $A_b$, bending area moment of inertia $I$, and length $L$. The position along the length of the beam is given by the variable $x$. The deflection of the beam is designated with the variable $y_d$. The nonlinear stiffness $k_{nl}$ is assumed to be a function of the tip displacement $k_{nl} = k_{nl}(y_d(L))$. 

Figure 34: Schematic of the nonlinear beam.


6.2 Ritz-Galerkin Discrete Model

A Galerkin approach was used to create a finite-order model of the experimental structure. Assuming that the beam behaves linear-elastically, mode shapes corresponding to transverse bending motion were used as shape functions to construct the Ritz-Galerkin representation [73]. The displacement of the beam at a position \( x \) was approximated as

\[
y_d(x,t) = \sum_{r=1}^{N_m} \psi_r(x) q_r(t)
\]

(70)

where \( \psi_r(x) \) is the \( r^{th} \) Euler-Bernoulli beam mode shape for a cantilever, \( q_r(t) \) is the \( r^{th} \) generalized coordinate, and \( N_m \) is the number of modes used. The system’s undamped equations of motion are provided in Eq. (71), where the coordinates are the amplitudes of the basis functions.

\[
\rho A_b L \ddot{q} + \frac{EI}{L} \dot{K} q = Q = \sum f_{ext} \psi_r(x_f)
\]

(71)

For the generalized coordinates, a time derivative is denoted with an over-dot (e.g. the generalized acceleration vector is \( \ddot{q} \)). Modal damping was added to the system by performing an eigenvector analysis on the linear system and then using,

\[
[C] = (\rho A_b L)^2 [M][\phi_b][\text{diag}(2\zeta, \omega_r)][\phi_b]^T [M]
\]

(72)

where \([\phi_b]\) is a matrix containing the eigenvectors in the columns, \( \omega_r \) is the \( r^{th} \) circular natural frequency, and \( \zeta \) is the \( r^{th} \) desired damping ratio. The generalized force vector \( Q \) is a sum of the product between all external forces and the value of the shape functions at the point where the force is applied, \( x_f \). Therefore, \( Q \) includes the applied or external forces, \( F_{ext} \) in Fig. 34, as well as the nonlinear restoring force due to the spring [73]. The beam provides linear stiffness at the
tip due to its flexural rigidity, so the discrete spring’s stiffness was chosen to be purely nonlinear as given below.

\[ k_{nl} = k_1 y_d(L)^2 \]  

(73)

The physical restoring force due to the spring is then equal to

\[ f_{sp} = k_1 y_d(L)^3. \]  

(74)

The generalized force vector then has components corresponding to the nonlinear spring \( \{ Q_{sp} \} \) located at \( x=L \) and the externally applied force \( \{ Q_{ext} \} \) located at \( x=x_f \).

\[ \{ Q \} = \{ Q_{sp} \} + \{ Q_{ext} \} = k_1 y_d(L)^3 \left\{ \begin{array}{c} \psi_1(L) \\ \vdots \\ \psi_N(L) \end{array} \right\} + F_{ext} \left\{ \begin{array}{c} \psi_1(x_f) \\ \vdots \\ \psi_N(x_f) \end{array} \right\} \]  

(75)

The external force is given by the following.

\[ F_{ext}(t) = A_{ext} \sin(\omega_f t) + A_d \sin\left(\frac{\pi}{\tau}(t-t_d)\right)(h(t_d) - h(\tau+t_d)) \]  

(76)

\( A_{ext} \) is the amplitude and \( \omega_f \) the frequency of the periodic forcing term that produces the limit cycle, while \( A_d \) is the amplitude of the disturbance forcing term that is used to perturb the system from the periodic orbit. The Heaviside step functions \( h(t) \) cause the impulsive force to initiate at \( t_d \) and to disappear after \( \tau + t_d \).

After using the Ritz-Galerkin method to form the discrete beam model and to account for the nonlinear applied force of the spring, the equations of motion were transformed back into physical coordinates using the relationship in Eq. (70). The differential equations of motion can then be arranged in state space format.
The matrix $\left[ \Psi \right]$ has the numerical values of the mode vectors for specific position coordinates on the beam. Since this matrix must be square in order to form an inverse, it is convenient to choose the number of degrees of freedom in the system to be equal to the number of modes used in the Galerkin expansion. Then, $\left[ \Psi \right]$ can contain shape functions evaluated at the nodal degrees of freedom on the beam. In this study the number of mode shapes used in the expansion and the number of degrees of measurement points (shown in Fig. 34) was $N=N_m=2$. The nodes were located at the center and tip of the beam as shown in Fig. 34.

In order to mimic the experimental system, the following parameters were used in the model, which are based on the nominal properties of the experimental hardware: $\rho=2700 \text{ kg/m}^3$, $E=68 \text{ GPa}$, $A_b=3.23*10^{-4} \text{ m}^2$, $I=4.34*10^{-9} \text{ m}^4$, $L=1.016 \text{ m}$. Using these properties with the Ritz-Galerkin method, the two linear natural frequencies of the system are $\omega_1/(2\pi)=9.97 \text{ Hz}$ and $\omega_2/(2\pi)=62.51 \text{ Hz}$. The tip spring in the model was based on the original experimental setup, which used spring steel, and the transverse stiffness is approximated in the model as $k_3=1.4764*10^9 \text{ N/m}^3$. A derivation of this approximation can be found in Appendix 3. It is important to mention that the experiment was designed to produce a cubic nonlinearity such as the one in the model. However, several practical design aspects could lead to other forms of nonlinearity in the response. For example, Fig. 33(a) shows that the nylon strip is offset from the neutral bending axis of the beam, which has the potential to produce a quadratic nonlinear
contribution to the response. Additionally, the fixed end of the cantilever is realized using bolts, and the effective length of the beam may change depending on the direction of the deflection. All of these issues could make the actual experimental hardware quite a bit more complicated than this simplified model.

6.3 Simulated Measurements

6.3.1 Periodic Orbits of the Nonlinear Beam

In order to apply the proposed nonlinear identification, the nonlinear beam must first be driven to respond in a periodic orbit. In Section 5.4, the Duffing system was used to show that one is able to exercise the nonlinearity most effectively when exciting either near resonance or at a superharmonic resonance. Either approach was found to work well, although the latter resulted in a more complicated linear time periodic system model, perhaps increasing the probability of making a mistake during the identification.

A multi-degree-of-freedom system could also be excited at a resonance or superharmonic resonance for each mode in the system, as well as at sum and difference frequencies, etc… In order to explore all of the possibilities, the numerical continuation technique described in Section 5.3 was again used to calculate the periodic orbits of the beam model for harmonic forcing with amplitude of \( A_{\text{ext}}=1 \) N and for forcing frequencies in the band 0.9-100 Hz. The results of the computation are shown in Fig. 35 with frequency response curves for the displacement and velocity initial conditions. Two dominant resonances are present in this bandwidth, and those are separated into the left (35(a) and (b)) and right (35(c) and (d)) columns of plots. The displacements in Fig. 35(a) are plotted with a logarithmic scale on the ordinate. The same conventions are used as in Fig. 16, but here there are two degrees of freedom, so the
displacement for the second degree of freedom is plotted in red and the velocity in black in order to distinguish them. The stable and unstable solutions are designated as before with solid and dashed lines, respectively. The displacements of both degrees of freedom peak sharply in 35(a) bending towards higher frequencies. The response in the second degree of freedom is larger since it is located at the tip of the cantilever, which is very active when the first resonance is excited. As the frequency increases, the displacements of both degrees of freedom approach the same value. The initial velocity of degree of freedom one in Fig. 35(b) is similar in shape to for the corresponding curves for the Duffing oscillator, but the curve for the second degree-of-freedom is more complicated, changing sign three times near the turning point.

The frequency response near the second mode is shown in Figs. 35 (c)-(d). The displacements of the different degrees of freedom have opposite signs near this resonance, as expected since this resonance is related to the second bending mode of the cantilever. Both curves in Fig. 35(c) have a peak that bends to higher frequencies, but there is only a very small region in the curves of the second resonance where the solutions are unstable. This second mode is less nonlinear than the first, as might be expected since displacements tend to become smaller with increasing frequency. The frequencies between 25 and 59 Hz were not shown; the response was essentially flat and had low amplitude in that region.
This system is more complicated than the Duffing oscillator, but its frequency response shows many of the same features. The hardening nonlinearity causes the peak in the first mode to bend from 15 to 22.5 Hz, to a final value that is more than double its linearized value, indicating that the response is highly nonlinear for this mode. Several sharp resonances are visible below 5 Hz (shown more clearly in Figure 36) each of which is a superhamonic
resonance of this dominant 22.5 Hz resonance. The largest of these superharmonic resonances has considerable amplitude, comparable to the primary resonance at 22.5 Hz. Two other superharmonic resonances are also visible in Fig. 35(a) near 13 and 21 Hz. These resonances show the displacement increasing for the first DOF while decreasing for the second. (The peak near 21 Hz in the red curve shows as a small gap where it becomes negative and cannot be plotted on a logarithmic scale.) These are superharmonic resonances for the second mode, as evidenced by the fact that these frequencies are about a fifth and a third, respectively of the frequency where the second dominant resonance peaks and also the fact that the displacements have opposite signs as do the peaks in 35(c).

The important implications of these nonlinear frequency response curves is that: 1) the dominant resonance peaks tend to bend towards higher frequencies for this system, which causes regions where multiple periodic orbits are possible for a single forcing frequency, and for this system one of those possible solutions is unstable (unstable solutions are designated in the plots with the dashed lines); and 2) there are clear superharmonic resonance peaks below 5 Hz for the first mode, and these designate conditions where large amplitude response is possible when the forcing frequency is a fraction of the linear resonance. The figure shows that one obtains the largest response when exciting near the 1st or second resonance.

The nonlinear frequency response was used to select a few forcing configurations near resonant peaks so that the periodic orbit would have large amplitude response. First, a forcing frequency of $\omega_r/(2\pi) = 15.4$ Hz was chosen on the first resonance peak of Fig. 35. Since the beam model has two modes that produce resonance responses, a second forcing frequency of $\omega_r/(2\pi) = 60.9$ Hz, which is just below the second resonance peak of Fig. 35, was chosen to explore the results of the identification when the periodic orbits from different modes are used.
For each case, the procedure that was used for the Duffing oscillator is used here to simulate the response and perform the identification. First, the acceleration periodic and periodic plus perturbation responses are simulated using MATLAB’s 4-5th order Runge-Kutta integrator function ‘ode45’. For each response, the amplitude of the harmonic forcing was \( A_{ext} = 1 \) N and the amplitude, duration, and time of initiation of the impulsive force were \( A_d = 10 \) N, \( \tau = 0.02 \) seconds, and \( t_d = 0 \) seconds (the impulsive force was set to zero when computing the periodic response). The response was calculated for a time history that was long enough for the perturbation to decay to 2.5% of its original value.

### 6.3.2 Forcing Case 1 (\( \omega_T/(2\pi) = 15.4 \) Hz):

The responses were calculated from initial conditions \( [y^T, \dot{y}^T] = [-3.517 \times 10^{-5}; -1.223 \times 10^{-4}; 3.639 \times 10^{-2}; 0.1160] \), which define the periodic orbit at 15.4 Hz, and the solution was sampled at a 616.4 Hz, which gave 40 samples per period of the periodic orbit. Figure 37(a) shows the first cycle of the response of the tip DOF. The periodic response, \( \bar{y}(t) \), is plotted with the dashed blue curve and the perturbed response, \( y(t) \), is plotted with the solid red line. The difference between \( y(t) \) and \( \bar{y}(t) \) was calculated which gives the approximate linear time periodic response, \( \tilde{y}(t) \), and that entire response is plotted in 37(b) with the solid black line. This plot shows that the perturbation is small compared to the periodic response and that it eventually decays to approximately zero, meaning the response has returned to the periodic orbit. The responses of the beam midpoint DOF are not shown, but they were in phase with the tip responses shown here and have similar characteristics but smaller extreme amplitudes.
Figure 37: Tip responses of the beam model for forcing frequency \( \omega_T/(2\pi) = 15.4 \text{ Hz} \).

The spectra of the responses were calculated with a Fast Fourier Transform (FFT) and are plotted in Fig. 38 for the tip degree-of-freedom (the time responses were sampled in order to avoid leakage for \( \omega_T \) and its harmonics). The responses, \( y(t) \), are plotted with the solid red lines and the perturbations, \( \tilde{y}(t) \), are plotted with the dashed black curves. The red curves contain a number of sharp peaks at the frequencies 15.4, 46.2, 77, 107.8, and 138.6, whose magnitudes diminish with increasing frequency. These frequencies correspond to the forcing frequency and a few of its odd harmonics (e.g. 3*15.4 Hz = 46.2 Hz). The spectra of the red and black curves also contain a number of broad peaks near 12.6, 18.2, 43.4, 49, 65.8, 74.2, 79.8, 96.6, 110.6, and 127.4 Hz in the tip DOF. The midspan DOF showed similar sharp peaks at forcing frequency harmonics and broad peaks at most of the previously mentioned frequencies.
The response of the beam is certainly nonlinear, since a single harmonic input was applied at 15.4 Hz, and the system responded at that frequency as well as at a number of its harmonics. The broad peaks in the response all seem to have the characteristics of linear mode peaks suggesting that this system has a very high order, however, many of these peaks are separated by integer multiples of the forcing frequency and are therefore evidence of time periodicity, arising because the response deviates slightly from the periodic orbit. For example, the peaks at 12.6 and 18.2 Hz have similar characteristics to the peaks at 43.4 and 49, and the two peak clusters are separated by 30.8 Hz, which is twice the fundamental frequency of the periodic orbit. These two peak clusters seem to be repeated at intervals of 30.8 Hz, although the magnitudes of the peaks change. The peaks at 65.8, and 96.6, and 127.4 Hz are also separated in frequency by 30.8 Hz and have similar characteristics albeit different magnitudes. The fact that many of these peaks seem to be related and occur at frequencies that can be linked by the fundamental periodic orbit frequency is strong evidence that a linear time periodic model is appropriate for $\ddot{y}_a(t)$. Noise or some other artifact in the response is not likely to have the
characteristic shape of a mode nor to occur at frequencies that are related by the periodic orbit frequency.

In order to fit a time periodic model to \( \ddot{y}_a(t) \), the lifting technique was applied to the responses and AMI was used to extract the modal parameters. The periodic response, \( \ddot{y}_a(t) \), was sampled 40 times per cycle, so the lifting technique produced 40 ‘lifted responses’ from \( \ddot{y}_a(t) \). The composite spectra (or average) of the lifted responses is plotted in Fig. 39 with the solid black line. The lifted spectrum contains one prominent peak near 2.8 Hz and a much weaker, broader peak near 4.2 Hz. All of the peaks in Fig. 38 have collapsed onto these two, indicating that only two modes have been excited. This justifies the use of only two measurement points when testing this system. AMI identified these two modes’ eigenvalues as \( \lambda_1 = -0.7310 +17.6158i \) and \( \lambda_2 = -3.9901 +26.2590i \). The corresponding natural frequencies are \( |\lambda_1|/(2\pi) = 2.806 \text{ Hz} \) and \( |\lambda_2|/(2\pi) = 4.227 \text{ Hz} \). A reconstruction of the two mode fit is plotted in the figure with the dashed line. The difference between the response curve and the fit is plotted with the dashed-dot gray curve, and it shows that the two-mode fit approximates the response very well.
Figure 39: Composite FFT of 40 pseudo-responses found by applying the lifting method to the linear time periodic response (solid black), reconstructed composite FFT identified by AMI (dashed red), and composite FFT of the difference between the simulated measurement and the reconstruction (dash–dot gray). Forcing frequency $\omega_T/(2\pi) = 15.4$ Hz.

In order to construct the state transition matrix using Eq. (24), the identified mode shapes must be expanded in a Fourier series, and then any spurious Fourier terms that result from measurement noise can be discarded. So, the identified modes were unaliased and were expanded in a Fourier series. Figure 40 shows a plot of the Fourier coefficient amplitudes for expansions of Mode 1 in 40(a) and Mode 2 in 40(b). The open blue circles and open green diamonds designate the $m=-10$ to $+10$ coefficients calculated in the expansion for DOF 1 and 2, respectively. However, not all of these coefficients (and also coefficients of higher $m$ value) are meaningful. In order to determine which were meaningful, the spectra of the perturbed minus the periodic response were plotted in 40(c) and interrogated to see which coefficients correspond to a peak. The same color scheme is used as with the Fourier coefficients, and the peaks corresponding to the modal eigenvalues ($m=0$ Fourier terms) with natural frequencies 18.2 and 65.8 Hz are labeled in the plot. Some peaks may be contaminated by other nearby modes or noise (e.g. the peak near 35 Hz in DOF 2 and 105 Hz DOF 1), so their correlation was
calculated with Eq. (26). After considering coefficients with a clear peak in the spectrum or with a peak having a correlation of at least 0.8, the peaks marked with filled markers in 40(a) and 40(b) were selected and designate the Fourier terms that were retained in Eq. (24) and used to compute $A(t)$.

![Fourier Series Expansion - Mode 1](image1)

![Fourier Series Expansion - Mode 2](image2)

![FFT of Simulated Responses](image3)

Figure 40: Fourier series expansions of the identified Mode 1 (a) and Mode 1 (b) for forcing frequency $\omega_T/(2\pi)=15.4$ Hz. (c) Approximate time periodic spectra. DOF 1 is designated in blue and DOF 2 in green.

For the first mode in Fig. 40(c), many coefficients have a large magnitudes and correspond to distinguishable peaks in the spectrum. For example, there are clear peaks at 49, 79.8, and 110.6 Hz in both degrees of freedom that can be attributed to the $m = +2$, +4, and +6
harmonics (i.e. 18.2+2*15.4 = 49, 18.2+4*15.4 = 79.8, and etc.). The peaks at 12.6 and 43.4 can be attributed to the $m=-2$ and -4 harmonics since for example, 18.2-2*15.4= -12.6 and negative frequencies reflect back to positive frequencies. The $m=-8$ term in the second degree-of-freedom in 40(a) must also be retained since there is a coherent peak at 105 Hz in the green curve in 40(c). There is some evidence of peaks at 74.2 Hz in the both degrees-of-freedom. A single mode was fit to these peaks and the correlation value was calculated with Eq. (26). For DOF 1, the correlation value is 0.75, which is less than the criteria of 0.8, so it was not retained, but the correlation value for DOF 2 is 0.85, so the $m=-6$ coefficient for DOF 2 was retained.

There are fewer dominant coefficients of the second mode in 40(b). One might presume from Fig. 40(c) that only the $m=0$ term is significant. However, there are coherent peaks in Fig. 40(a) at 35 Hz in DOF 1, at 96.6 Hz in both DOFs, and at 127.4 Hz in DOF 2, all of whose shapes suggests that they are not noise artifacts. These peaks correspond to the $m=-2$, +2, and +4 harmonics of the expansion for the mode at 65.8 Hz. The green curve has a small bump in the curve at 35 Hz, but the correlation value of this peak is 0.74, so the $m=-2$ coefficient for DOF 1 was not retained. Similarly, the peak in the blue curve at 127.4 Hz has a correlation factor of 0.68, so its coefficient was not retained.

After the Fourier series models of the identified modes have been formed, they were integrated twice according to Eq. (32) in order to obtain a displacement description of the modes. Then, the state transition matrix and state coefficient matrix were constructed based on the method described in Section 3.2.3. The system is fourth order, so $A(t)$ is a $4\times4$ time dependent matrix. The lower left $2\times2$ block corresponds to scaled stiffness components of the model as shown in Eq. (52). The (3, 2) and (4, 2) components of the state coefficient matrix, which are the stiffness between midspan-to-tip and tip-to-ground, respectively, are plotted versus time in
Figure 41 with open circles. An analytical model for $A(t)$ was also constructed using the equations of motion, and its components are plotted with a solid line. The estimated values of these coefficients agree well with the analytical values. The (3, 2) term differs from the analytical term by at most about 3.9% and the (4, 2) term differs by at most about 8%. Both of the terms shown change significantly over the periodic orbit, the (3, 2) term by approximately 30% and the (4, 2) term by approximately 100% of their initial values, revealing that the model for this system (linearized about the periodic orbit) is certainly time-periodic. The stiffness is largest at the instants where the displacement $\overline{y}_a(t)$ is maximum, suggesting that the nonlinearity is of the hardening type.

![Figure 41](image-url)

**Figure 41**: Time periodic state coefficient matrix components for forcing frequency $\omega T/(2\pi)=15.4$ Hz. Open circles are the values estimated from the simulated measurements, while the solid lines give the values computed from the analytical model.

The (3, 1) and (4, 1) terms of the state coefficient matrix are also related to the stiffness of the system, but for brevity they are not shown. The analytical (3, 1) and (4, 1) terms do not
vary with time, but their estimates showed spurious variation in time of between 3% and 17%, respectively. However, they were relatively accurate in an average sense.

Because the damping in the system is linear, the lower right 2×2 block of the true analytical state coefficient matrix is also composed of terms that are constant with time. The estimates of these damping terms from the identification method were found to vary significantly with time. Each of the four terms varies by an amount that is at least an order of magnitude larger than the constant value predicted by the analytical model, and they did not tend to agree very well in an average sense either. Like the results from the Duffing oscillator, the error seems to arise when estimating $A(t)$ from the state transition matrix. Artificial noise was not added to any of the simulated measurements discussed in this section, so they are theoretically exact except for the errors that the time-integration routine introduces due to its finite step size. The errors that have been observed in the simulated system identification seem to arise because the time periodic model that has been identified from the measurements contains only the dominant Fourier coefficients, $N_B$. One would expect that perfect results would be obtained if all of the Fourier coefficients were identified, i.e. if $N_B \to \infty$. Lastly, $A(t)$ was also estimated from Fourier terms that had a correlation of at least 0.9 and then 0.7. When the stricter metric value of 0.9 was used, the differences in the $A(t)$ coefficients from those in Fig. 41 was negligible. However, when the metric was relaxed to a value of 0.7, the certain terms (e.g. the $A(3,2)$) were significantly different from the analytical results. So the correlation value of 0.8 seems to lead to Fourier terms that provide the best possible $A(t)$ for this case.

The identified state coefficient matrix can now be used to determine the nonlinear accelerations (or restoring forces) in the equations of motion using Eqs. (47) and (54). This was done using MATLAB’s ‘cumtrapz’ function, which approximates the cumulative integral using
the trapezoidal method. As mentioned in Section 3.4.1, there is an unknown constant of integration in each term. These were accounted for by removing the mean of each term in \[ f^i(x) \], causing each curve to pass through the origin.

In order to verify the identified model for the nonlinear accelerations, Eq. (54) can be used to calculate \( \{g(x_d, x_o)\} \), which is the total restoring acceleration in the system at each instance within the periodic limit cycle. The restoring acceleration can also be computed from the modified restoring force surface method described in Section 3.4.2.2, and the two can be compared to validate the results. (One could instead use Eqs. (55) and (56) to calculate and compare the restoring forces.) The total restoring acceleration acting on a DOF is the sum of components due to the stiffness of the beam and dissipative effects of damping. Figure 42 shows this sum with open circles plotted versus time for one cycle of the periodic orbit. The accelerations computed using Eq. (56) with the mass matrix \( M_p \) from the model are shown with the solid line. In the first half of the orbit, the restoring acceleration on the beam tip leads the restoring acceleration on the beam center, which initially remains near zero, but both curves are in phase and have a somewhat sinusoidal shape, with DOF 1 being more heavily distorted than DOF 2. The total restoring accelerations that were identified with the NL-LTP method agree exceptionally well with those calculated with the modified restoring force method. However, the damping terms in the estimated \( A(t) \) contained many errors, but those values were several orders of magnitude smaller than the stiffness terms. So, it is likely that those forces related to damping contribute much less to the net-nodal restoring accelerations, and the method provided here does not necessarily verify that the identified nonlinear acceleration-velocity relationships are correct.
Figure 42: Time periodic restoring accelerations acting on the beam degrees of freedom for forcing frequency $\omega_T/(2\pi)=15.4$ Hz.

Now the individual restoring accelerations computed with Eq. (47) can be explored. The left-$2\times2$ block of $[g^x(x)]$, corresponds to the acceleration-displacement relationships of the nonlinear system, each of which is plotted with open blue circles in Fig. 43. The top row of plots relates to the restoring accelerations experienced by the first DOF and the bottom row to those felt by the second DOF. The analytical acceleration-displacement relationships were also formed and are plotted with solid lines. The black curves and black markers pertain to a different forcing frequency that will be presented subsequently. The underlying periodic orbit defines the range of displacement (horizontal axis) for each DOF. The first DOF displaces between about $\pm0.5$mm (43(a) and 43(c)), while the second DOF displaces farther, between about $\pm1.2$mm (43(b) and 43(d)). The restoring acceleration on the first DOF ($g^x_{1,1}$) is negative with positive displacement of $y_{d,1}$, indicating that positive displacement $y_1$ causes a negative acceleration $a_{1,1}$, returning $y_{d,1}$ to equilibrium. This is also seen for the $g^x_{2,2}$ versus $y_{d,2}$ relationship, while the opposite is true for both of the cross terms, $g^x_{1,2}$ and $g^x_{2,1}$ (positive
displacement of $y_{d,2}$ causes $y_{d,1}$ to accelerate in a positive direction). The magnitudes of the $[g_{1,1}^x, g_{1,2}^x]$ and $[g_{2,1}^x, g_{2,2}^x]$ acceleration components oscillate between about $\pm 50 \text{ N/kg}$ and $\pm 90 \text{ N/kg}$, respectively. All of the acceleration-displacement curves are predominantly linear, except for the $g_{2,2}^x$ component in (d), which reflects the $\approx 100\%$ stiffness change that was noted in Fig. 40(b). All of the identified acceleration-displacement relationships agree very well with the analytical relationships.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure43.png}
\caption{Acceleration-displacement relationships acting on the beam DOFs for simulated measurements from forcing frequencies $\omega_f/(2\pi)=15.4 \text{ Hz and } 60.9 \text{ Hz}$.}
\end{figure}

6.3.2.1 Discussion

The acceleration-displacement curves can be directly linked to the dynamics of the system and how the DOFs interact. These functions could be used to predict the free response of the system for an arbitrary initial condition, so long as the displacements of DOF 1 and DOF 2 do not exceed the ranges shown in Figure 43. According to these results, the nonlinearity is only significant between the spring at the tip and ground, and this is consistent with analytical model.
As for the Duffing oscillator, these results were obtained without any a priori assumption regarding the shape of the force-displacement relationships or the location of the nonlinear spring. The damping terms that were identified in the $A(t)$ matrix were found to be significantly in error, however, their contribution to the total restoring forces was small. Also as for the Duffing oscillator, one would have to use some other method if nonlinearities in the damping terms were of primary interest.

6.3.3 Forcing Case 2 ($\omega_T/(2\pi)=60.9$ Hz):

For the second case, the forcing frequency is $\omega_T/(2\pi)=60.9$ Hz, which is near the second linear resonance condition, and the initial conditions for the periodic orbit are $[\bar{y}_d^T, \bar{y}_v^T]=[-8.631e-5; 1.217e-4; 9.103e-2; -1.345e-1]$. The same procedure as with Case 1 is used to perform the identification. AMI identified two modes in the responses with eigenvalues $\lambda_1= -3.9850 +11.5712i$ and $\lambda_2= -0.6290 +72.8480i$ and natural frequencies $|\lambda_1|/(2\pi)= 1.9478$ Hz (62.8 Hz when unaliased) and $|\lambda_2|/(2\pi)=11.5945$ Hz, respectively.

The responses for this forcing case contain weaker time periodic effects than in the previous case, and thus the Fourier series expansion models of the mode shapes are quite straightforward to construct. The Fourier series were formed for each mode shape, and the $m=-2$, 0, and +2 terms were retained for the 11.6 Hz mode ($\lambda_2$ from AMI) for both degrees of freedom and the $m=0$ and +2 terms were retained for the 62.8 Hz mode ($\lambda_1$ from AMI) for both degrees of freedom. The state coefficient matrix, $A(t)$, was constructed from the Fourier series model with the same procedure as Case 1, and was found to vary only weakly with time (much less so than that shown in Fig. 41 for Case 1). For example, the $(4, 2)$ coefficient varied by about 11%
(compared to 100% for Case 1). Once again, the identified stiffness terms agreed well with the analytical model but the damping terms had significant errors.

Next, the restoring accelerations were calculated. The left-2×2 block of the resulting function \( g^x(x) \) is plotted overlaid on the previous results in Fig. 43 for the estimated and analytical acceleration-displacement relationships using open black squares and solid black lines, respectively. The acceleration-displacement curves for Case 2 compare exceptionally well with those from Case 1, although the displacements of the DOFs for this periodic orbit are much smaller than the previous case, and correspondingly the extreme magnitudes of the restoring accelerations are also smaller. This causes the nonlinearity to be much less visible.

6.3.3.1 Discussion

When this system was excited with a frequency near the second nonlinear resonance, a completely different periodic orbit was produced. The responses of the two measurement points were out of phase, and the magnitudes of the displacements were smaller. Despite these differences, the procedure was still very straightforward to apply, and an accurate time periodic model was identified. Due to the smaller amplitudes of the DOFs, the time periodic effects identified in the model were weaker, and the acceleration-displacement relationships that were constructed were only weakly nonlinear. Nevertheless, the acceleration-displacement relationships did agree well with those identified in Case 1, at least over the range in which they had been identified. The Case 2 result could also be used to simulate the response of the system, but only over about half the range of each DOF that one could use with the result from Case 1. These results suggest that this new identification method is not all that sensitive to the frequency of excitation or the mode that is excited. However, the excitation must exercise the nonlinearity over the full range of interest in order to obtain a useful model of the system. For this system,
higher forcing amplitude must be used to exercise the nonlinearity when exciting near the second resonance.

6.4 Experimental Measurements

The NL-LTP identification method was next applied to the experimental system. Harmonic excitation was applied to the beam with a model 2100E11-100 lb Modal Shaker from The Modal Shop, Inc. The beam was approximated as having a fixed support, so the shaker was freely hung from a lateral excitation stand, as recommended in [74]. A thin steel stinger was used to transmit the excitation from the shaker to the beam. One end of the stinger was clamped inside the shaker armature and the other end was fixed to a force transducer, model 208C04 from PCB Piezotronics, Inc. (PCB), which was bolted to the beam at a location $x = 45$ mm from the fixed end of the beam. Harmonic forcing was provided by a Tektronix model 3022 arbitrary function generator, amplified by a model 2050E05 power amplifier from The Modal Shop, Inc., and input to the modal shaker. The amplitude and frequency of the excitation were varied to establish different forcing configurations. A modally tuned impulse hammer, model 086C01 from PCB was used to provide the impulsive force that perturbed the response from its periodic orbit. The response was measured with two Endevco model 66A12 triaxial accelerometers (z-channels were used) located at $x = 45$ mm (DOF 1; shaker location) and $x = 508$ mm (DOF 2; beam center) and with an Edevco model 256-100 isotron accelerometer located near the free end of the beam at $x = 984$ mm (DOF 3; beam tip). The degree-of-freedom locations are labeled with arrows in Figure 32. All the accelerometers were secured to the beam with wax. The measurements were acquired using a Photon II portable dynamic signal analyzer by LDS Dactron.
6.4.1 Experimental Forcing Case 1: \((\omega I/(2\pi) = 44 \, \text{Hz})\):

The function generator was used to drive the system at 44 Hz with maximum amplitude of approximately 60 N, which excited the beam near the second bending resonance and produced tip displacements of a few millimeters. The impact hammer was used to apply an impulsive force at \(x = 375 \, \text{mm}\) which excited the three modes in the response, and the force transducer signal and three accelerometer signals were sampled at a frequency of 2.56 kHz over a window of approximately 12.8 seconds. The sampling frequency provided measurements at 60 samples per cycle of the periodic orbit, and the time window allowed twice the time that would be needed for the response to return to the periodic orbit. In order to determine the underlying periodic orbit and match its phase to that of the perturbed responses, the second half of the time histories, which were dominated by periodic orbit alone, were used to estimate \(\bar{y}_a(t)\). The exact frequency of the periodic orbit was found by fitting a multi-harmonic sinusoid to the periodic orbit using MATLAB’s ‘fminsearch’ function. This frequency was found to be 44.00 Hz, and the multi-harmonic sinusoid that was found was used to recreate the periodic orbit over the entire time series. The reconstructed periodic orbit \(\bar{y}_a(t)\), the perturbed response, \(y_a(t)\), and the difference between the two, \(\tilde{y}_a(t)\), are plotted for the tip response (DOF 3) in Fig. 44 with the same format that was used previously. The perturbation about the periodic orbit can clearly be seen in the early time response, since it fluctuates with higher frequencies about the periodic response. Note also that the periodic orbit is far from a pure sinusoid (as one would expect for a linear system) but rather looks triangular. The other degrees of freedom had opposite phase, since the system is responding near a second bending mode, and DOF 2 (midspan) had a similar shape periodic orbit.
while DOF 1 had a highly distorted shape with periodic high frequency fluctuations. The response of DOF 1 is an order of magnitude lower, since it is near the bolted base of the beam.

![Graph showing initial and approximate LTP tip responses](image-url)

*Figure 44: Tip responses of the experimental beam for forcing frequency $\omega f/(2\pi) = 44\, \text{Hz}$.*

FFTs of the perturbed response and the perturbed minus the periodic responses were calculated, and the spectrum for the tip response is plotted in Fig. 45 using the same format that was used previously. A number of sharp peaks occur in the perturbed response at 44 Hz and all of the harmonics of that frequency (i.e. 88, 132, 176 Hz, etc.) that are shown in the plot. The 44 Hz peak was also dominant in the other response degrees of freedom (not shown) and the remaining peaks did not substantially diminish with increasing frequency in those DOF either. Since a single 44 Hz frequency sinusoid was used to force the system, this suggests that the system is highly nonlinear. There is a good deal of noise in the signal and some artifacts that are not readily explained, but several coherent peaks can be seen that appear to be caused by time periodicity, for example at 17.35, 26.65, 42.6, 45.4, 61.35, 70.65, 120.35 Hz, etc… The peak at 17.35 Hz is separated from the peak at 61.35 Hz by the drive frequency, 44 Hz. The peak at 45.4
Hz is separated from the peak at 133.4 Hz by twice the drive frequency. All of those peaks as well as others could be evidence of time periodic effects.

![Response spectra at the tip of the experimental beam for forcing frequency $\omega_T/(2\pi)=44$ Hz.](image)

Following the procedure, the lifting method was applied to the perturbed minus periodic responses, and the FFT of the lifted responses was calculated. The composite spectrum of the lifted responses is plotted in Fig. 46 with the solid black line. Three peaks can be seen in the lifted spectrum, and AMI was able to identify a coherent mode at each of those peaks. After removing those modes from the spectrum it appears to contain a mostly constant noise floor level and some artifact near the zero frequency line. The eigenvalues identified by AMI were $\lambda_1 = -2.418 + 8.858i$, $\lambda_2 = -2.805 + 73.234i$, and $\lambda_3 = -0.2606 + 108.95i$ and natural frequencies $|\lambda_1|/(2\pi) = 1.461$ Hz, $|\lambda_2|/(2\pi) = 11.664$ Hz, and $|\lambda_3|/(2\pi) = 17.341$ Hz, respectively.
Figure 46: Composite FFT of 60 pseudo-responses found by applying the lifting method to the linear time periodic response (solid black), reconstructed composite FFT identified by AMI (dashed red), and composite FFT of the difference between the measurement and the reconstruction (dash–dot gray). Forcing frequency $\omega_{f}/(2\pi) = 44$ Hz.

The lifted responses in Fig. 46 were certainly far easier to interpret than the spectrum in Fig. 45. However, once the eigenvalues have been identified in the lifted spectrum it is straightforward to see which eigenvalue has caused each of the peaks in the standard spectrum in Fig. 45. The dominant peaks in Fig. 45 are assumed to be the modes of the system. Using this assumption, the first eigenvalue can be deciphered to be an aliased version of the mode near 45.4 Hz, since $1.46 + 44 = 45.46$, the second eigenvalue is the aliased version of the mode near 120.35 Hz, since $11.664 - 3 \times 44 = -120.336$, and the third eigenvalue was not aliased and is the mode near 17.35 Hz. The time periodic modes that AMI extracted from the measurements were expanded in a Fourier series, and since the modes were somewhat noisy, the perturbed minus periodic response spectra were carefully interrogated to determine which peaks correspond to actual Fourier coefficients and which to noise. The correlation factor was calculated and used in this process as well, as was done in Section 6.3.
Figure 47 shows the amplitudes of the Fourier series coefficients plotted versus coefficient number for the three modes. All the coefficients for $m=-6...6$ are plotted with open markers. The degrees of freedom are distinguished with blue circle, green diamond, and red square markers for DOFs 1, 2, and 3, respectively. Figure 48 shows the spectra of the time periodic responses for DOF 1 (blue line), 2 (green line), and 3 (red line). The peak corresponding to each modal eigenvalue is labeled in the plot. The peaks that produced each of the Fourier coefficients in Fig. 47 can be found by adding integer multiples of 44 Hz to each modal frequency. The Fourier terms that were retained are plotted with filled markers. Some of these were selected without the aid of the correlation metric. For example, the response at DOF 3 shows a clear peak at 70.65 Hz in Fig. 48 so it was retained (this is the $m=-2$ harmonic of Mode 1 in Fig. 47(a)), and similarly for the 133.4 Hz peak (the $m=+2$ term for Mode 2 in Fig. 47(b)). These peaks did not necessarily have high correlation values, presumably due to noise or distortions in the measurements, but they are clearly meaningful.
Figure 47: Fourier coefficients of the three modes identified from the measurements for forcing frequency $\omega_T/(2\pi) = 44$ Hz. Open markers signify all the coefficients and filled markers signify coefficients that were retained to form $A(t)$. Circle, diamond and square markers were used for DOF 1, 2 and 3 respectively.
Inspection was not reliable for many of the other peaks, so their correlation values were used to determine whether to retain them. For example, there is a small artifact at 164.35 Hz in Fig. 48, which would be a $m=+1$ term for Mode 3 in Fig. 47(c) (i.e. $120.35+1*44=164.35$). The correlation of this peak in DOF 1 (blue curve) is 0.85, so that coefficient was retained, but the correlation for DOF 3 (red curve) is 0.784, so that coefficient was omitted. ($A(t)$ was calculated with and without this term since its correlation was close to the criteria, and its effect was found to be negligible). Finally, DOF 2 had no meaningful terms for Mode 3, perhaps because the response at DOF 2 is near a node for the third linear mode.

Next, the Fourier series model based on the retained terms was integrated twice, used to form the state transition matrix for the system, and then $A(t)$ was calculated. The terms related to stiffness in rows four, five, and six and column three of the state transition matrix are plotted versus time in Fig. 49(a). The measured tip displacement over one cycle of the periodic orbit is plotted in 49(b). The identified $A(t)$ matrix is highly time varying. For example, the $A(4,3)$, $A(5,3)$, $A(6,3)$ terms vary by about 43%, 92%, and 48%, respectively, from their initial values over the course of one period, suggesting that the stiffness changes by as much as 92% over the periodic orbit. The maximum tip stiffness occurs when the tip displacement is maximum,
suggesting that the nonlinearity is a hardening type. However, the tip stiffness increases to a slightly larger value (i.e. more negative) when the tip displacement is negative (rather than positive). This might be caused by the imperfect constraint at the base of the cantilever or by the offset of the nylon strip from the beam's neutral bending axis.

![Figure 49: Estimated time periodic state coefficient matrix components and tip displacement for forcing frequency $\omega_f/(2\pi)=44$ Hz.](image)

The identified damping terms in the state coefficient matrix fluctuated considerably, as was also the case for the simulated measurements. The estimated damping terms seemed to be very sensitive to the Fourier terms used in the construction of $\Phi(t,t_0)$ and $A(t)$, and their magnitudes were several orders smaller than the stiffness terms, so at this time the model for damping does not seem to be trustworthy. On the other hand, it is possible that the damping in this experimental system would actually vary somewhat over each cycle.

The state coefficient components were next integrated according to Eq. (47) in order to calculate the individual restoring accelerations acting on each DOF. In order to first assess the
validity of the identified $A(t)$ matrix and the restoring force relationships, the total restoring force acting on each node was calculated as it was in the previous section using Eq. (54) and Eq. (55). The mass matrix required for the latter method was generated with Eq. (57) using linear modes that were extracted using a low-level hammer excitation test, as discussed in Section 3.4.2.2. This matrix was slightly ill-conditioned (its singular values were 7.13, 0.14, and 0.015), so Eq. (55) was used to validate the identified model in terms of restoring forces instead of the accelerations, since this calculation does not require the inverse of the mass matrix. The same mass matrix was used for both restoring force calculations, and the forces are plotted in Fig. 50, with the open circles showing the forces from the NL-LTP method and the line showing the forces from the modified restoring force surface method. The forces for DOF 1 and 2 ((a) and (b)) are out of phase with that of DOF 3 since the system is excited near a second bending mode. The results for both methods agree quite well, suggesting that the identified model is reasonable.

Figure 50: Total restoring forces acting on measured degrees of freedom for forcing frequency $\omega_T/(2\pi)=44$ Hz.
Next, the individual restoring accelerations in $[g^x(x)]$ can be interrogated. The acceleration-displacement relationships are plotted in Fig. 51. The extreme values of the periodic displacement can be assessed by looking at the abscissa of each plot; they are largest for the measurement point nearest to the tip. Some noise is visible in each term and some terms show some slight hysteresis, which is most likely spurious. Many of the terms seem to be predominantly linear, except for those in the third column, which show clear signs of cubic nonlinearity. This column was the result of integrating the time varying $A(4,3)$, $A(5,3)$, and $A(6,3)$ components. (One should bear in mind that this type of plot does not accentuate the nonlinearity. Figure 49 clearly shows that the stiffness changes significantly with displacement amplitude.)

Figure 51: Estimated acceleration-displacement relationships for experimental system with forcing frequency $\omega_T/(2\pi)=44$ Hz.
6.4.1.1 Discussion

The experimental beam was significantly more complicated than the Ritz model. Three modes were excited in the linear time periodic response, and there were many harmonics as well as noise complicating the measured spectra. The nonlinear acceleration-displacement (or alternatively force-displacement) relationships were identified from real measurements on actual hardware, without having to make any a priori assumptions regarding their form. One can fit a polynomial to these functions to find a more convenient model for the nonlinearity in the system. For example, a 3\textsuperscript{rd}-order polynomial was fit to the $g_{2,3}^x$ and $g_{3,3}^x$ functions, and the following models were obtained:

$$g_{2,3}^x(y_3) = 0.33y_3^3 + 0.26y_3^2 + 17.75y_3 - 1.12$$

and

$$g_{3,3}^x(y_3) = -0.82y_3^3 + 0.0047y_3^2 - 62.96y_3 - 1.07.$$  These functions contain small, but significant cubic terms, and the fit of the $g_{2,3}^x$ had a significant quadratic term. The presence of quadratic terms might be due to the offset of the nylon strip from the beam's neutral bending axis. The range of the identified nonlinear models might be increased if a higher forcing amplitude is used, however, the measurement noise seemed to increase drastically with applied forces larger than 60 N.

6.4.2 Experimental Forcing Case 2: ($\omega_T/(2\pi)=14.3$ Hz):

A second forcing frequency was used in order to check the consistency of the NL-LTP method, and to attempt to exercise the nonlinearity more fully. The beam was forced at $\omega_T/(2\pi)=14.3$ Hz, which was found to produce a more complicated periodic orbit, as shown in Fig. 52. The acceleration of DOF 1 (blue solid line) is quite small compared to that of DOF 2
(dashed green line) and DOF 3 (dotted red line), and all of the responses contain a certain amount of noise. There is also at least one higher order harmonic in the periodic response.

Figure 52: Reconstructed periodic response over one cycle for forcing frequency $\omega_f/(2\pi)=14.3$ Hz.

The NL-LTP procedure was performed using the same steps as with the previous cases. For brevity, many of the details are omitted here. In summary, three time periodic modes were identified in the perturbation response, $\ddot{y}_a(t)$, and their natural frequencies were 14.97, 46.32, and 120.06 Hz. The Fourier series model was formed using the meaningful terms for each mode and was used to construct $\Phi(t,t_0)$ and $A(t)$. Then, the latter matrix was integrated to calculate the acceleration relationships.

Figure 53 shows the (6,3) component of $A(t)$, which relates the stiffness acting on the tip DOF, as well as the estimated displacement of the tip. The stiffness in 53(a) is time varying and is more complicated than what was found in the previous case (Fig. 49). Additionally, the displacement of the tip in 53(b) is far different from a pure sine wave; the tip displaces to its maximum value and then seems to hold that position for a short time. Once again the stiffness increases when the tip displacement is maximum or minimum, but the stiffness increase is quite
different in each case. One might think that this is evidence of asymmetry in the system caused by nonlinear effects, but it seems more likely that the rest of the beam is exerting a complicated time varying force on the beam over the course of the limit cycle, which comes about because the periodic orbit is so complicated.

![Graph showing time periodic state coefficient matrix component (6,3) and tip displacement for forcing frequency $\omega_f/(2\pi) = 14.3$ Hz.](image)

**Figure 53:** Estimated time periodic state coefficient matrix component (6,3) and tip displacement for forcing frequency $\omega_f/(2\pi) = 14.3$ Hz.

In order to check the identified acceleration-displacement model, the forces that were estimated using $\omega_f/(2\pi) = 14.3$ Hz were plotted against the results of the previous forcing case. Figure 54 shows the identified $g_{3,3}$ terms for $\omega_f/(2\pi) = 14.3$ Hz with the black line with square markers and the terms for the previous forcing case with the blue line. The inset plot shows the detail of the curves for the maximum negative displacement. The linear terms of the curve agree quite well, but there are some small errors at the extreme displacements where largest different between the two curves (at maximum positive displacement) is 2.9%. Many of the other terms were also plotted and show similar agreements with some differences. The largest difference was approximately 15% between the $g_{1,3}$ terms. This term was mostly linear, so the difference is
likely due to noise in the measurements and inconsistencies in the constant terms of the Fourier
descriptions of the identified modes. Considering all the possible sources of noise in the
experimental setup and the large difference in the periodic orbits that were used for the
identification, it is encouraging to see how well the curves in Fig. 54 agree.

Figure 54: Estimated acceleration-displacement relationships for DOF 3 from different forcing
frequencies, $\omega_T/(2\pi) = 14.3$ and $44$ Hz.

6.4.2.1 Discussion

This second forcing case produced a considerably more complicated periodic response
than the first. The periodic orbit was not a simple sinusoid, but contained higher harmonics;
forcing at this frequency seemed to excite the second resonance (at approximately $3\omega_T$) as well
as the first. The NL-LTP procedure was still straightforward to apply, but the estimated state
coefficient terms were more complicated and could not be readily interpreted by simply
comparing them with the limit cycle displacements. However, both forcing cases activated the
nonlinearity over similar ranges and led to similar estimates of the acceleration-displacement
relationships.
6.5 Summary

In this section, the NL-LTP method was verified for simulated measurements of a two-degree-of-freedom cantilever beam and was then applied for the first time to actual measurements from a beam (a continuous system) with a nonlinear spring at its tip. In both cases the nonlinear force-displacement relationships governing the motion of each system were identified without having to assume the form of the nonlinearities a priori. In each case two different excitation frequencies were used and the nonlinear models found in each case produced consistent results.

In the last step of the method, the linearized equations of motion are integrated to recover the acceleration-displacement and acceleration-velocity relationships of the underlying nonlinear system. These are related to the system’s nonlinear force-displacement and force-velocity relationships by the mass matrix. One can use these to plot the force-displacement or acceleration-displacement curves for each of the degrees of freedom of the system. These plots provide significant insight, as illustrated by the examples in the preceding section, especially when dealing with a system that has a complicated limit cycle (e.g. Figures 53 and 54).

For the multi-degree-of-freedom beams, the correlation metric helped to discern the peaks in the spectrum that are physically meaningful from those that are might be dominated by measurement noise. The examples presented here showed that one can readily estimate the modes of the linear time periodic model from a set of measurements, but not all of the modal parameters are equally trustworthy. Here this issue has been addressed by discarding any terms that were likely to contain significant errors and the correlation metrics presented here were found to be very helpful in this regard.
7 Conclusions

This dissertation presented a new experimental identification technique for nonlinear systems which was named the Nonlinear-Linear Time Periodic (NL-LTP) method. The method is based on approximating the system's motion about a stable limit cycle with a linear time periodic model that describes the linear dynamics of the system near the nonlinear periodic orbit. In the last step, the linearized equations of motion are integrated to recover the underlying nonlinear system for all the instances contained in the periodic orbit. Furthermore, the resulting nonlinear model can also be used to compute the total restoring forces acting on the measured degrees-of-freedom. The key advantages of the method include: 1) the method is applicable to multi-degree-of-freedom systems; and 2) the order of the system and the form of the nonlinearities are identified without assuming these characteristics a priori.

Several tools were developed in order to facilitate the application of the NL-LTP method to higher order nonlinear systems.

- A correlation metric was developed to help validate the estimated state transition matrix. The state transition matrix is built from the Fourier series model, and the correlation metric ensures that only physically meaningful terms are retained, while terms that are due to measurement noise are discarded.

- A method was developed to integrate the estimated $A(t)$ matrix in order to estimate the nonlinear force-displacement relationships.

- A validation metric was used to verify the total restoring forces acting on the measured degrees-of-freedom. This total restoring force metric is based on the restoring force surface method, which is widely known.
To help select the forcing configuration at which to apply the identification, a pseudo arc-length continuation technique was developed to calculate the nonlinear frequency responses of harmonically forced nonlinear systems. A single analysis with this method can capture all of the resonances (i.e. superharmonic, hardening, softening etc…) and stability information that describe all the periodic orbits of the system within a frequency band, and this can be used to select the optimal harmonic forcing parameters to apply the NL-LTP identification method.

The NL-LTP method was applied to noise-free, simulated measurements of a Duffing oscillator and two degree-of-freedom nonlinear cantilever beam. For each system, the NL-LTP method was then applied to a variety of different periodic responses (selected from the computed nonlinear frequency responses). The nonlinear acceleration-displacement models were accurately identified and the models were in agreement for different forcing cases. The identified model is limited to the range exhibited by the periodic orbit, but large amplitude displacements can be excited if the forcing frequency is near the frequency of a dominant resonance or superharmonic resonance. The acceleration-velocity models that were identified contained significant errors. However, the damping in the example systems was very small. Therefore, the damping must be significant in order to be identified with the NL-LTP approach, and some other method may be needed to identify damping models for systems with small damping parameters. The Duffing system was also used to explore the effects of the perturbation. The best results were achieved when the perturbation was as small as possible, while still resolved above the measurement noise. That being said, good experimental results were achieved in this work for perturbations of 0.5% and 2%.

The NL-LTP method was also applied to experimental measurements from a high order nonlinear system. The nonlinear beam was notably more complicated than the model and the
measurements contained a significant amount of noise. Despite these challenges, the procedure was still very straightforward to apply. The correlation metric was especially helpful, since there were Fourier expansions for each mode in the response and at each degree-of-freedom. The nonlinear restoring accelerations were estimated and verified for the force-displacement relationships. Similar to the model, the estimated damping terms were small and not trustworthy.

It is interesting to consider the parallels between this method and modal analysis of linear time invariant systems. In both cases the modes of the state transition matrix are clearly manifest in the frequency response and can be readily identified using similar techniques (for an LTI system $A(t)$ and $\Phi(t, t_0)$ share the same modes). The goal of conventional modal analysis is often simply to estimate the modes of the system; one typically does not attempt to use the identified modes to estimate specific terms in the mass or stiffness matrices of the system. In contrast, in this work, the modes of the linearized (linear time periodic) system model are not of particular interest, but are sought because they can be used to find the nonlinear restoring forces between each of the degrees of freedom of the system. It would probably be far easier to instead compare the identified linear time periodic modes with those predicted by a computational model, and then to update the computational model so that it produces linear time periodic modes that match those that were identified experimentally. In any event, there are disadvantages to that approach and this work provides a useful alternative.

8 Future Work

Based on the evidence provided in this dissertation, once you have a periodic orbit, the Nonlinear-Linear Time Periodic Identification method is quite simply to apply. The major
challenges seem to be: 1) finding a periodic orbit that does not contain nonlinear phenomena like severe period doubling; 2) keeping the perturbation about the periodic orbit small but measurable; and 3) selecting appropriate terms to use to construct $A(t)$ so that it is accurate. In this section, some ideas will be provided for future work on the NL-LTP method. This will include a discussion on the third point mentioned above as well as a discussion of applying the identification method to a nonlinear gait system.

8.1 **Improving the Time Periodic Model Estimation Process**

A major limitation of the NL-LTP method is the sensitivity of the estimated time periodic state coefficient matrix, particularly for terms that are related to damping, to the Fourier terms that are used to construct the state coefficient matrix. For all the systems considered in this dissertation, the estimated damping terms had significant errors, and these errors seemed to arise when estimated the $A(t)$ matrix from the state coefficient matrix. This procedure needs to be explored to see whether any improvements can be made or whether there is an alternative procedure that can be used to estimate the $A(t)$ matrix.

The identification may perform better in its current form if the system of interest exhibits high levels of damping (linear and potentially nonlinear). Larger damping values may be more easily identified if they can be clearly resolved from the measurement noise. So, the NL-LTP method could be explored for systems exhibiting high levels of damping.

8.2 **Applications to Nonlinear Human Gait**

Recently, the dynamics of the human musculoskeletal system has been a research topic of interdisciplinary interest. In this system, muscles with nonlinear force-displacement relationships apply forces to joints, and the joints rotate through large angles, so the response is
nonlinear. Gait simulations of the human musculoskeletal systems [75, 76] are providing important new understanding of the dynamics of the human body that could have a large impact on clinical applications of gait [77, 78], such as improving treatments for gait disorders. However, there are still many uncertainties in these simulations and in the biomechanical models, so experiments are important. The system identification approach studied in this work could help to extract more information from gait measurements, but some form of the forced response presented in Section 3.3 must be used because one might not want to initiate the impulse response of the gait motion of a human subject in experiment because they could fall.

A high order musculoskeletal model such as that used in [75] could be used to evaluate the applicability of the identification method on experimental human gait measurements. These models are driven by a set of muscle actuations that are computed by the Computed Muscle Control (CMC) algorithm [75] in order to reproduce the motion of actual human subjects. The first objective would be to determine whether the modal parameters of a linear time periodic approximation to a system such as this are sensitive to changes in the musculoskeletal model. This could be evaluated by computing the analytical linear time periodic modes, a few of which will likely dominate. Then, after a small change is made to one of the model parameters (such as a change in the muscle actuations of two opposing muscles), the analytical linear time periodic modal parameters will be recomputed. If the change in model parameters results in a change in the dominant computed linear time periodic modes, then the proposed identification techniques of this work are likely to be viable for use on experimental gait measurements.

Additionally, a simpler inverted pendulum walker model could also be studied. The model consists of two massless, rigid links of length $l$ connected with a potentially damped hinge. This is shown in Fig. 55. A concentrated mass $M_p$ is located at the hinge. Two additional
concentrated masses, \( m \), are located at the free end of each of the links. The model is located in space by the coordinate of one foot mass, and then the orientation is defined by the leg angles \( \theta_1 \) and \( \theta_2 \), where positive rotation is taken to be counterclockwise. The force of gravity is present, and the vector valued (in \( \mathbb{R}^2 \)) reaction forces \( \{ F_1 \} \) and \( \{ F_2 \} \) are time dependent functions that cause the model to walk periodically. Some simulations have already been performed on a model that is similar to this one.

![Nonlinear inverted pendulum walker model](image)

**Figure 55: Nonlinear inverted pendulum walker model.**

The inverted pendulum walker model is nonlinear due to the large angle rotations of the pendulum legs. In order to walk stably, the model relies on impulsive ground reaction forces, which are discontinuous, so the walker may not lend itself to a linearized model. It is a useful model because the dynamics of this simple model are quite representative of some of the dynamics seen in experimental measurements of human walking. The model could be used to simulate gait measurements in order to verify the NL-LTP method for the forced response approach, since it has yet to be applied. This will likely pose a new set of challenges to understand, so that the method can still be applied successfully.
A1 Appendix 1: Saddle Node and Pitchfork Bifurcations

In some simple cases, one can establish a functional relationship between a changing parameter and the changing dynamics of a nonlinear system. This idea is explored in the next two sections for systems that produce a saddle node bifurcation and a pitchfork bifurcation.

A1.1 Saddle Node Bifurcation

Let $U$ be an open subset of the real numbers, $\varepsilon \in \mathbb{R}$ be a small parameter, and $f$ be the $C^1$, one-dimensional vector field that determines the following differential equation.

$$\dot{x} = \varepsilon + x^2 = f(x, \varepsilon)$$

(78)

The equilibria $\bar{x}$ for the differential equation are $\bar{x} = \pm \sqrt{-\varepsilon}$. There are three regions of interest that dictate the dynamics of the one-dimensional system: $\varepsilon < 0$, $\varepsilon = 0$, and $\varepsilon > 0$. The function $f(x) = \varepsilon + x^2$ is plotted in Fig. 56 for the different regions of interest since it is just the equation of a parabola. The locations of the roots (if applicable) are designated with a filled circle. The number of roots the system contains changes as the parameter $\varepsilon$ changes from negative to positive. The system starts with two unique equilibria. When $\varepsilon = 0$, the system changes configuration so that it has a repeated equilibrium at the origin. Finally, the system has zero equilibria when $\varepsilon > 0$. 
Figure 56: Curves of the function \( f(x) = \varepsilon + x^2 \) for variation of the small parameter \( \varepsilon \) as negative (a), zero (b), and positive (c). The filled circles locate the roots of the parabola which are also the equilibria for the system.

For the first case of \( \varepsilon < 0 \), the linearizations about the equilibria are

\[
\begin{align*}
\frac{\partial f}{\partial x} \bigg|_{-\sqrt{-\varepsilon}} &= -2\sqrt{-\varepsilon}, \text{ stable} \\
\frac{\partial f}{\partial x} \bigg|_{\sqrt{-\varepsilon}} &= 2\sqrt{-\varepsilon}, \text{ unstable}
\end{align*}
\]

where the stability is determined by the sign of the eigenvalues of the linearizations [34, 35, 37]. Since no eigenvalue is zero, the system is hyperbolic and said to be structurally stable, meaning that it has dynamics that are qualitatively the same in the presence of small perturbations to the differential equation [34]. When the parameter \( \varepsilon \) is zero, there is only one equilibrium and its linearization for the repeated root.

\[
\frac{\partial f}{\partial x} \bigg|_{-\sqrt{-\varepsilon}} = 0
\]

In this configuration the system is not hyperbolic due to the zero eigenvalue. If a small perturbation is applied to \( \varepsilon \), the system could bifurcate into a configuration having two unique equilibria or zero equilibria. Clearly, the bifurcation value of this system is \( \varepsilon = 0 \). The final
configuration is the case of $\varepsilon > 0$ has no equilibria. It is also structurally stable against small perturbations in this configuration [34].

A bifurcation diagram is helpful for understanding the changes in the equilibria of the system with the changes in the parameter of interest. The bifurcation diagram for this saddle node bifurcation is shown in Fig. 57 [79]. The stable portions of the curve are plotted with a solid line and the unstable portions with a dashed line. By taking a vertical slice of this diagram, one obtains the equilibrium points for the system. From Fig. 57, one can see how the two equilibrium points converge to a single repeated equilibrium when negative $\varepsilon$ moves to zero. For $\varepsilon$ greater than zero, the equilibria vanish.

![Bifurcation diagram](image)

**Figure 57:** Bifurcation diagram for $\dot{x} = \varepsilon + x^2$ for variation of the small parameter $\varepsilon$.

Using the results in Fig. 57, one can begin to understand the cyclic fold bifurcation that occurred in the response of the DO {Hale, 1991 #127;Seydel, 1994 #194}. Recall that the DO was initiated in a periodic orbit that turned out to be unstable, and the response diverged from its initial trajectory to a new trajectory that appeared to be a different periodic response. At least qualitatively, Fig. 57 shows that a dynamic configuration allowing two equilibrium states is possible. One can imagine starting in a configuration on the dashed curve in Fig. 57 and
diverging towards the stable configuration. In fact, saddle node bifurcations are very common in systems, so a lot can be gained from the simple qualitative analysis performed here.

### A1.2 Pitchfork Bifurcation

Another very typical bifurcation that is commonly seen in practice is the pitchfork bifurcation. Consider the differential equation

$$\dot{x} = \varepsilon x - x^3$$

where as before $x \in U \subset \mathbb{R}$ is the one-dimensional vector field defined on an open subset $U$ of the real numbers and $\varepsilon \in \mathbb{R}$ is a small parameter. The equilibrium $x$ for the differential equation are $\bar{x} = 0, \pm \sqrt{\varepsilon}$. The curves of the function $f(x) = \varepsilon x - x^3$ are provided in Fig. 58.

![Figure 58: Curves of the function $f(x) = \varepsilon x - x^3$ for variation of the small parameter $\varepsilon$ as negative (a), zero (b), and positive (c). The filled circles locate the roots of the parabola which are also the equilibria for the system.](image)

In this case, negative $\varepsilon$ causes a configuration that has a single equilibrium at the origin, $\varepsilon = 0$ leads to a triple equilibrium point at the origin, and $\varepsilon > 0$ causes a configuration with three unique equilibrium. As with the saddle, the linearizations can be calculated, and this also provides the stability.
The system is structurally stable when \( \varepsilon \neq 0 \), since the eigenvalue of these cases (\( a \) and \( c \) in Equation 82) since they do not have eigenvalues of zero. When \( \varepsilon < 0 \), the origin is a stable equilibrium. When \( \varepsilon > 0 \), the origin is an unstable origin and the values of \( x = \pm \sqrt{\varepsilon} \) are both stable equilibria. The case of \( \varepsilon = 0 \) is not structurally stable. A small perturbation could either inject two stable equilibria and one unstable equilibrium into the system, or create one stable equilibrium. Therefore, \( \varepsilon = 0 \) is a bifurcation value for the system. The bifurcation diagram for this system is shown without proof in Fig. 59.

\[
\begin{align*}
  \frac{\partial f}{\partial x} &< 0, \text{ stable; } \varepsilon = 0, \text{ nonhyperbolic; } \varepsilon > 0, \text{ unstable} \\
  \frac{\partial f}{\partial x} &< 0, \text{ stable; } \varepsilon > 0, \text{ unstable} \\
  \varepsilon > 0 &< 0, \text{ stable} \\
\end{align*}
\]

Figure 59: Bifurcation diagram for \( \dot{x} = \varepsilon x - x^3 \) for variation of the small parameter \( \varepsilon \).

Although the analysis presented here can be used for higher dimensions, the results get very complicated quickly. Guckenheimer and Holmes [34] and Hale and Koçak [79] both present analysis of bifurcations in higher dimensions and include some helpful diagrams. The proofs of existence and creation of the curves shown in bifurcation diagrams such as Fig. 57 and
59 are provided in [35]. Abraham and Shaw take a purely visual approach to explaining the qualitative analysis of bifurcations in [80].

A2 Appendix 2: Calculating Nonlinear Frequency Responses

A2.1 Introduction

Many techniques have been developed to calculate the periodic solutions of both forced and undamped autonomous nonlinear systems, most of which take one of two basic approaches. The first approach uses analytical techniques to find an approximate solution to the differential equations. The method of multiple scales and the method of averaging are some of the most commonly used perturbation techniques that find approximate solutions for the nonlinear response by varying some small parameter. A number of applications of these methods can be found in the following works [17, 81, 82]. The harmonic balance technique is a popular approach where one assumes a harmonic series for the solution and usually truncates the series after a few terms [17, 68]. Many terms must be included in the expansion if the nonlinearity is strong, making the mathematical formulation more difficult. While these methods are very valuable, in some cases the small parameter assumption and the truncated series are not valid, and even when they are the methods are not easy to automate for a general system and require a certain level of expertise to be sure that the approximations are warranted. The second type of approach is based on numerical methods. If a system is damped then one can always calculate the periodic response by integrating the equations of motion from an arbitrary set of initial conditions until the system settles into a steady periodic orbit. Commonly used time integration routines include the adaptive time step control, fourth-order Runge-Kutta method and the Newmark method. The numerical approach is powerful because closed form equations of
motion are not required. However, this direct approach tends to be too computationally expensive, so methods have been developed over the years to streamline such calculations. In fact, methods that find the periodic solutions of unforced systems (homogenous differential equations) are somewhat mature [67]. Two typical approaches are to evaluate and update a cost function with optimization [83, 84] or to solve a boundary value problem using a Newton-Raphson correction method [67]. The latter approach is widely used because the Newton-Raphson method converges quadratically. These approaches are usually started with an initial guess, which is subsequently corrected to satisfy convergence criteria, and this overall procedure is generally termed a *shooting* method. A parameter of interest can then be varied and the shooting method applied multiple times to compute a branch of solutions, and this is termed a *continuation* method. A general overview of shooting and continuation techniques is provided in [69].

Generally speaking, the methods for continuation can be categorized as corrector techniques, predictor-corrector techniques, or adaptive predictor-corrector techniques. Correction techniques use the general shooting method to calculate solutions. Continuation is commonly performed with these methods by incrementing the forcing frequency (perhaps sequentially), using the previous solution as the initial conditions for the shooting method calculation of the solution. This procedure is improved by using a predictor-corrector algorithm, where each solution is used to calculate a prediction for the next solution on the curve. A prediction can be made for both the new forcing frequency and the new periodic response values. Then, the prediction is used as the initial guess for some form of shooting method that calculates the actual solution, and the convergence is usually much faster than without a prediction. Predictions can be made with a number of different geometric calculations, for example by
calculating the tangent to the branch of solutions. More advanced methods will use an adaptive predictor-corrector algorithm, where the number of convergence calculations are recorded and used to adapt the prediction increment size between successive solutions, which further improves the efficiency of the calculations. A few well developed packages exist that use various forms of adaptive predictor-corrector methods for response and bifurcation analysis. AUTO [32] is a well developed Fortran® code for autonomous nonlinear systems. MATCONT [85] is a Matlab® based code with some similar capabilities. Neither of these packages is intended to solve forced response problems, but apparently they have been adapted to do this at times by adding degrees of freedom for a harmonic oscillator and coupling it to the nonlinear system. The author was unable to find this documented in the technical literature but has heard experts mention this possibility.

Some of these techniques have been adapted to forced response problems. In [86], a continuation method was used to calculate the forced-response of the midspan of a beam with clamped ends that was modeled with a Galerkin finite element model. The forcing frequency was varied by a simple secant prediction and the previous response solutions values were used as the initial guess for the correction portion of the method. The method showed promising results, but the algorithm could not follow sharp turns of the periodic solution branch. In [87] and [88], the harmonic balance method was used with adaptive predictor-corrector algorithms to calculate the periodic solutions of a reduced order torsional subsystem for the former and to calculate the periodic solutions of three degree of freedom model of a dual gearbox-shaft system for the latter. For each of these works, the harmonic balance method was used to assume truncated series solutions to the equations of motions, which created a set of algebraic equations to solve for branches of solutions with the continuation method. This approach produces very detailed
periodic solution curves but still requires one to use the approximations required by the harmonic balance method. If too few terms are used in the expansion then the algorithm may not predict important nonlinear features, as noted in [87]. A form of predictor-corrector continuation was also used in [89, 90]. They incremented the forcing frequency sequentially and then calculated a prediction for the state vector with a first order Euler expansion from the previous solution. The algorithm provides quite robust performance. For example, in regions where a periodic solution branch has a turning point (i.e. a multiple solutions region), the algorithm included a re-parameterization of the solution curve in order to follow the turning point. The algorithm was applied to a few low order systems and showed promising results. More recently, a similar form of predictor-corrector continuation was presented in [33]. This method employs a pseudo-arc length continuation procedure that has a tangent predictor step and orthogonal corrector steps. Furthermore, the algorithm automatically controls the step size and exploits symmetry in the response to increase computational efficiency. So far, the authors of this method have computed the periodic solutions of a number of unforced, undamped nonlinear systems of high complexity and high order [16, 33, 91-95] with exceptional detail. However, all of those works are concerned only with the free response of the nonlinear system.

This appendix presents a continuation method that efficiently calculates the periodic solutions of harmonically forced nonlinear systems. The approach used here is similar to that employed by [33], although with several important modifications to account for the forcing function. A Newton-Raphson correction technique will be used to calculate the first solution from an initial guess that the user must supply (e.g. a low frequency solution from a static analysis). In order to calculate the subsequent solutions, first an adaptive predictor step is used that varies both the frequency and the state vector, and then correction steps are calculated that
employ a second Newton-Raphson formulation to find the next value on the nonlinear frequency response function. The process continues until the desired frequency band has been characterized.

Section A2.2 contains an outline of the proposed continuation method followed by a review of the shooting method and Newton-Raphson correction methods. The necessary changes of those procedures due to the harmonic forcing will be derived. The continuation procedure will then be explained in detail.

**A2.2 Theory**

A nonlinear system with input $u(t)$ can be described by the following state space differential equation

$$
\dot{z} = f(z(t), u(t))
$$

(83)

where $z \in \mathbb{R}^n$ is the time dependent state vector, $u \in \mathbb{R}^p$ is a time dependent input vector, and $t$ is the time variable. For structural dynamic systems the state variable $z$ is the vector valued function of time $z = [x^T \; \dot{x}^T]^T$ composed of the $N$ displacement degrees of freedom (DOF) $x \in \mathbb{R}^N$ and the $N$ time derivatives of the displacement DOF, $\dot{x} \in \mathbb{R}^N$, where $n=2*N$. The superscript ‘T’ denotes the matrix transpose operator. It is assumed that $f$ is a $C^1$ (at least one-time continuously differentiable) nonlinear function defined on an open subset of $\mathbb{R}^n$ that contains all the possible displacements and velocities of the system of interest. For a specific input, the vector field $f$ generates the flow $Z_t \in \mathbb{R}^n$ that contains families of solution curves of the differential equation [34]. A single solution curve is defined by its initial state vector $z_0$ and input vector $u_0$ and by tracing the flow $Z_t(z_0)$ for a finite time.
This work considers systems with harmonic forcing so \( \ddot{u}(t + T) = \ddot{u}(t) \) and hence \( \dot{z} = f(z, T) \), where the dependence of the system on the fundamental period \( T \) is explicitly shown. When the system reaches steady state, it is assumed that the flow contains at least one periodic orbit \( \gamma \) with period \( T \), and \( \bar{z}(t + T) = \bar{z}(t) \) for any state \( \bar{z} \in \gamma \). (From here forward, the overbar will be dropped to simplify the notation.) Depending on the configuration of the system and its parameters, including the forcing frequency and amplitude, the flow may contain multiple periodic orbits [34, 63]. Tracking the periodic orbits in a nonlinear system can be difficult because a small change in the parameters of the system, such as the forcing frequency, may drastically change the dynamics of the flow. For example, a stiffening system has a frequency response peak that bends to higher frequencies, and the response can jump from a high amplitude response to a very low amplitude response if a perturbation is introduced to the system or as the frequency is increased beyond the critical value. In order to characterize all of the periodic orbits, one must calculate all of the initial state vectors \( z_0 \) that produce a periodic orbit for each \( T \).

This work presents a continuation method that calculates all of the periodic orbits along a branch, resulting in an estimate of the nonlinear frequency response over some range of frequency. The general procedure is outlined below.

1. **Provide a starting guess:** \((z_0^{(0)}, T^{(0)})\)
   - The user supplies \((z_0^{(0)}, T^{(0)})\) where the superscript zero in parentheses designates that this is the initial guess that is used to start the method.

2. **Calculate a starting solution:** \((z_0^{(1)}, T^{(1)})\)
   - The proposed continuation method must be started from an actual periodic orbit of the system. To calculate this first solution from the starting guess, \((z_0^{(0)}, T^{(0)})\),
a Newton-Raphson correction method (NRCM 1) is applied, which accounts for changes in both the state vector $\Delta z_0$ and the period $\Delta T$. After a certain number of iterations a converged solution, $(z_{0,(1)}, T_{(1)})$, is obtained where the subscript number in parentheses designates the converged solution number.

3. **Initiate the predictor-corrector procedure to calculate the next solution along the branch.** The last solution found is denoted with a $j$, as $(z_{0,(j)}, T_{(j)})$, so in this step the $(j+1)^{th}$ solution is sought:

   a. **Prediction Step:** $(z_{0,(j+1)}, T_{(j+1)}^{(0)})$
      
      o The current solution is used to calculate a prediction $(z_{0,(j+1)}, T_{(j+1)}^{(0)})$ for the next periodic solution.

   b. **Correction Step:** $(z_{0,(j+1)}, T_{(j+1)}^{(k)})$, $k = 1, 2, 3, \ldots$
      
      o The prediction step is used as the initial guess, $(z_{0,(j+1)}, T_{(j+1)}^{(0)})$, for the second Newton-Raphson procedure. The second Newton-Raphson correction method (NRCM 2) is similar to NRCM 1, while also constraining the corrections to be orthogonal to the prediction. At the end of iteration number $k$, NRCM 2 has produced an updated estimate of the parameters, $(z_{0,(j+1)}, T_{(j+1)}^{(k)})$, for the $(j+1)^{th}$ point along the branch. Then convergence is assessed and:
      
      o **If a solution is obtained:** Step 3 is repeated to find the next point, i.e. $(j+2)$, along the branch.

      o **If a solution is not obtained:**
1. If the correction number $k$ is within some pre-set number of iterations, then the current correction, $(z^{(k)}_{0,(j+1)}, T^{(k)}_{(j+1)})$, is used as an initial guess for the NRCM 2 method to calculate the $(k+1)^{th}$ correction, $(z^{(k+1)}_{0,(j+1)}, T^{(k+1)}_{(j+1)})$, similar to Step 3b. The convergence is assessed again.

2. If the convergence criteria are not met within some pre-set number of iterations: The procedure calculates a new prediction, $(z^{(0)}_{0,(j+1)}, T^{(0)}_{(j+1)})$, using a smaller step and Step 3b is restarted.

The details of this procedure will be provided below, including the derivation of the general Newton-Raphson correction method. Some parts follow the derivation from [33], but there are several important differences that arise due to the forcing that will be noted.

### A2.3 Shooting Function

The proposed method employs an iterative process to find the periodic response of a system for a certain forcing frequency. The *shooting function* is created to quantify whether a certain set of initial conditions, $(z_0, T)$, produces a periodic response. First, the forcing frequency is set to $\Omega=2\pi/T$ and Eq. (83) is integrated over the interval $(0, T)$ from an initial state $z_0$ until $t=T$ at which point the state is denoted $z_T$. Then, the shooting function $H \in \mathbb{R}^n$ is defined as follows.

$$H(z_0,t,T) = z_T(z_0,t,T) - z_0 \quad (84)$$

Both $z_T$ and $H$ have three arguments. The second argument in each is the time at which $z_T$ or $H$ is evaluated while the third argument is the period of the forcing frequency that is driving the system. Equation (84) represents a two-point boundary value problem for the periodic orbit of
the system, which is satisfied when $H(z_0, T) = 0$. In practical numerical computation, the shooting function will be satisfied when $H(z_0, T) \approx 0$ for some chosen convergence criterion. In this work, the convergence criterion defined in [33] is used

$$\frac{\|H(z_0, T)\|}{\|z_0\|} = \frac{\|z_T - z_0\|}{\|z_0\|} < \epsilon$$  \hspace{1cm} (85)$$

where $\epsilon = 1e-6$ is the convergence value. The shooting function is calculated each time the equations of motion are integrated over a period in order to monitor convergence.

### A2.4 Newton-Raphson Correction Methods

When the given calculation of the shooting function does not satisfy the convergence criterion, a Newton-Raphson [67] method is used to calculate updates $\Delta z_0$ and $\Delta T$ to the initial conditions. Following the approach used in [33], the shooting equation $H(z_0, t = T, T) = 0$ can be expanded in a Taylor series about $(z_0 + \Delta z_0, T + \Delta T)$,

$$H(z_0, t = T, T) + \frac{\partial H}{\partial z_0}(z_0, T) \Delta z_0 + \frac{\partial H}{\partial T}(z_0, T) \Delta T = 0,$$  \hspace{1cm} (86)$$

where only the linear terms have been retained. Some subtleties regarding these derivatives must be noted. The first partial derivative, $[\partial H/\partial z_0] \in \mathbb{R}^{n \times n}$, has the same form as in [33], and can be expanded as

$$\frac{\partial H}{\partial z_0}(z_0, T) = \frac{\partial z(z_0, t, T)}{\partial z_0} \bigg|_{t=T} - [I]$$  \hspace{1cm} (87)$$
where \([I]\) is the \(n \times n\) identity matrix. The time varying Jacobian matrix \([\partial z / \partial z_0] \in \mathbb{R}^{nxn}\) is the variation of the state vector at time \(t\) due to a small change in the initial conditions. It can be calculated using one of two methods described in detail in [33]. The first method, which uses a finite differences approach, can be used even when the equations of motion are not available in closed form, but it is more computationally expensive and is not used in this work. To implement the second method, the following system of time varying differential equations is formed by differentiating Eq. (83) with respect to \(z_0\), switching the order of differentiation on the left hand side, and applying the chain rule on the right hand side. The result is,

\[
\frac{d}{dt} \left( \frac{\partial z(z_0,t,T)}{\partial z_0} \right) = \left[ \frac{\partial f(z,t,T)}{\partial z} \right]_{z=z(t)} \left[ \frac{\partial z(z_0,t,T)}{\partial z_0} \right]
\]

(88)

where the Jacobian matrix \([\partial f / \partial z] \in \mathbb{R}^{nxn}\) is simply the linearization of \(f\) about the state \(z(t)\) for each time instant \(t\) in the interval \((0, T)\). Since, the state vector was calculated over the interval \((0, T)\) in order to evaluate the shooting function, it can be easily stored in order to integrate the system of Eq. (88) from initial conditions \([\partial z(z_0,t,T)/\partial z_0] = [I]\). The solution matrix \([\partial z/\partial z_0]\) can then be evaluated at \(t = T\) and inserted into Eq. (87) to obtain \([\partial H/\partial z_0]\).

The vector \(\{\partial H/\partial T\} \in \mathbb{R}^n\) in Eq. (86) differs from the result presented in [33], since the addition of the forcing function makes \(f\) explicitly dependent on both \(t\) and \(T\). This partial derivative can be expanded using the chain rule as

\[
\frac{\partial H}{\partial T} = \left. \frac{\partial H}{\partial t} \right|_{t=T} + \left. \frac{\partial H}{\partial T} \right|_{t=T}
\]

(89)
where the first term represents the change in the shooting function due to a change in the instant that is taken to be the end of the period and can be calculated from Eq. (83) as in [33].

\[
\frac{\partial H}{\partial t} \bigg|_{t=T} = f(z_T, t = T, T) \quad (90)
\]

The second term in Eq. (89) was not present in [33], but arises here because the forcing function causes \( f(z, t, T) \) to be dependent on the period of the forcing, \( T \). This term represents the change in the shooting function due to a change in the period (or frequency) of the input.

\[
\frac{\partial H}{\partial T} \bigg|_{t=T} = \frac{\partial z(z_0, t, T)}{\partial T} \bigg|_{t=T} \quad (91)
\]

Note that the second term in the shooting function does not appear in either of these expressions because the initial conditions do not depend on the period, \( \{\partial z_0/\partial T\} = \{0\} \). Using the same approach that was used to derive Eq. (88), one can derive the following system of time varying differential equations that can be solved to calculate \( \{\partial z/\partial T\} \).

\[
\frac{d}{dt} \left( \frac{\partial z(z_0, t, T)}{\partial T} \right) = \left[ \frac{\partial f(z, t, T)}{\partial z} \right]_{z=z(t)} \left( \frac{\partial z(z_0, t, T)}{\partial T} \right) + \frac{\partial f}{\partial T} \quad (92)
\]

This set of equations contains a forcing term \( \{\partial f/\partial T\} \in \mathbb{R}^n \) which arises due to the harmonic input force. Hence, this second set of differential equations can be integrated from zero initial conditions \( \{\partial z(z_0, t, T)/\partial T\} = \{0\} \) over the interval \((0, T)\) in order to compute \( \{\partial z(z_0, t, T)/\partial T\} \).

This completes the computation of all of the terms in Eq. (86), which now constitutes a linear system of equations that can be solved for an update \( (\Delta z_0, \Delta T) \) to the initial conditions.
The details are slightly different at different steps in the continuation algorithm, as explained below.

**A2.5 Continuation Procedure**

**A2.5.1 Provide a starting guess: \((z^{(0)}_0, T^{(0)})\)**

In order to start the continuation procedure, the user must supply a starting guess. A good starting point guess \((z^{(0)}_0, T^{(0)})\) consists of a low frequency solution for the underlying linear system (i.e. for \(T^{(0)}\) the underlying linear system can be solved for the periodic conditions \(z_0^{(0)}\)). This is often called a Homotopy approach [17, 67].

**A2.5.2 Calculate a starting solution using NCRM 1**

The starting guess is used to calculate a set of initial conditions \((z_0, T)\) that leads to a shooting function, \(H(z_0, T)\), that satisfies convergence. To calculate the starting solution, the system is integrated from the starting guess, the shooting function is calculated, and the convergence is evaluated. If the initial guess does not satisfy convergence, then the partial derivatives are calculated and the following system is formed to calculate corrections to both the state vector \(\Delta z_0\) and the period \(\Delta T\). This system is the NRCM 1 mentioned above.

\[
\begin{bmatrix}
\frac{\partial H}{\partial z_0}(z_0, T) & \frac{\partial H}{\partial T}(z_0, T)
\end{bmatrix}
\begin{bmatrix}
\Delta z_0 \\
\Delta T
\end{bmatrix} = \{-H(z_0, t, T)\}
\]

(93)

The right hand side vector contains the vector value of the shooting equation. The new initial conditions are \((z_0^{(1)} = z_0^{(0)} + \Delta z_0, T^{(1)} = T^{(0)} + \Delta T)\). Since the period has been updated the harmonic forcing frequency must also be updated in the equations of motion (i.e. \(\Omega_{update} = 2\pi/(T^{(1)})\)) such that \(u = u(\Omega_{update})\). Then, these new conditions are used to integrate the
equations of motion and check the shooting function for convergence. One then repeats this two-stage process until the desired convergence is achieved, at which point the first periodic orbit solution \((z_{0(1)}, T_{(1)})\) has been calculated. The Newton-Raphson corrector method is a local technique, so the starting guess, \((z_{0}^{(0)}, T^{(0)})\), must be close to the actual solution or the correction scheme may not converge. The first periodic orbit solution and the Jacobian matrices that were used to produce it can be used to initiate the next step.

A2.5.3 Initiate the predictor-corrector procedure to calculate a branch of solutions:

\((z_{0(j)}, T_{(j)}), j=1,2,3,4,...\)

In order to calculate an entire branch of solutions, an adaptive pseudo arc-length continuation [69] method is employed that uses a prediction step which is followed by a correction step. Predictor-corrector methods can follow complex solution branches because the prediction step follows the direction in which the branch of solutions actually evolves. The prediction step, which is calculated using information from the current periodic solution, is used to calculate new initial conditions for the next periodic orbit. The prediction step can be further adapted to evaluate the convergence of the previous correction steps and adjust the step size accordingly. After the prediction step, the shooting method is employed, but now with a new Newton-Raphson correction method that varies the state vector and the forcing frequency in a constrained direction.

A2.5.4 Prediction Step: \((z_{0,(j+1)}, T^{(0)}_{(j+1)})\)

The basic process is to use the previous branch solution, which is denoted the \(j^{th}\) solution \((z_{0(j)}, T_{(j)})\), to calculate a tangent predictor step for the \((j+1)^{th}\) solution. Both the initial state vector and the forcing period are allowed to vary when computing this prediction. The prediction \(\{P\} \in \mathbb{R}^{n+1}\), which has components \(\{P_{2}\} \in \mathbb{R}^{n}\) and \(P_{T} \in \mathbb{R}\) corresponding to the states
and period respectively, is the tangent vector to the solution branch and can be calculated with
the following system of equations.

\[
\begin{bmatrix}
\frac{\partial H}{\partial z_0(z_{0,(j+1)}T_{(j+1)})} & \frac{\partial H}{\partial T(z_{0,(j+1)}T_{(j+1)})}
\end{bmatrix}
\begin{bmatrix}
P_{(j)}
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]  \hspace{1cm} (94)

The components of the left hand side matrix come from the Taylor series expansion of
the shooting function, and were previously defined. They characterize the vector space that is
normal to the periodic solution curve, and once they are found, the desired tangent vector \{P\} is
simply the vector that is in the null space of the matrix on the left in Eq. (94). Once the tangent
vector \{P\} has been calculated it can be normalized to unit length, and the predictions for the
next periodic solution can be calculated with

\[
z_{0,(j+1)}^{(0)} = z_{0,(j)} + s_{(j)} P_{z,(j)}
\]  \hspace{1cm} (95)

\[
T_{(j+1)}^{(0)} = T_{(j)} + s_{(j)} P_{T,(j)}
\]  \hspace{1cm} (96)

\[
\Omega_{(j+1)}^{(0)} = 2\pi / T_{(j+1)}^{(0)}
\]  \hspace{1cm} (97)

where \(z_{0,(j+1)}^{(0)}\) and \(T_{(j+1)}^{(0)}\) are the predicted values for the \((j+1)^{th}\) solution and \(s_{(j)}\) is the current step
size, i.e. the step size at the \(j^{th}\) point along the branch. One must be sure to again update the
differential equations so that the external forcing reflects the change in frequency (i.e.
\(u = u(\Omega_{(j+1)}^{(0)})\)).

The step size is critical to the success and efficiency of the computation. One can ensure
an accurate result by fixing the step size to a very small value, but this would waste
computational effort at many points along the branch. A better approach is to use an automated
step size control algorithm. In this work, the step size control algorithm from [33] is employed.
The first step size is provided by the user, while all subsequent step sizes are determined, based
on the number of iterates that were required to update the previous shooting function solution,
denoted $K_{(j-1)}$, and the target number of shooting iterates, $K^*$, which is supplied by the user.

$$s(j) = sgn\left(s_{(j-1)}\left\{P_{(j)}\right\}^T\left\{P_{(j-1)}\right\}\right)\left[\frac{K^*}{K_{(j-1)}}\right]s_{(j-1)}$$

(98)

The signum function $sgn(*)$, ensures that the step used to calculate the predictions follows the
solution curve in the same direction. Furthermore, it is helpful to place maximum and minimum
bounds on the step size.

**A2.5.5 Correction Step NCRM 2: $\left(z_{0,(j+1)}, T_{(j+1)}^{(k)}\right)$, k =1,2,3,...**

As discussed in [33], the Newton-Raphson updates to the shooting functions can be made
more efficient by constraining the corrections to be orthogonal to the tangent predictor vector.
This is achieved by computing the correction steps with the following algebraic system, which is
the second correction method, NRCM 2, mentioned previously.

$$\begin{bmatrix}
\frac{\partial H}{\partial z_0} \left(z_{0,(j+1)}, T_{(j+1)}^{(k)}\right) & \frac{\partial H}{\partial T} \left(z_{0,(j+1)}, T_{(j+1)}^{(k)}\right) \\
\{P_{z,(j)}\}^T & P_{T,(j)}
\end{bmatrix}
\begin{bmatrix}
\Delta z_{0,(j+1)}^{(k)} \\
\Delta T_{(j+1)}^{(k)}
\end{bmatrix}
= \begin{bmatrix}
-H \left(z_{0,(j+1)}, T_{(j+1)}^{(k)}\right) \\
0
\end{bmatrix}$$

(99)

All of the terms in this equation have been defined previously. The solution to this
equation provides the $k^{th}$ corrections $\Delta z_{0,(j+1)}^{(k)}$ and $\Delta T_{(j+1)}^{(k)}$ to the guess for the $(j+1)^{th}$ solution.
The algorithm then alternates between a shooting function calculation $H(z_{0,(j+1)}, T_{(j+1)}^{(k)}) \approx \{0\}$ and
a correction step calculation of $z_{0,(j+1)}^{(k+1)} = z_{0,(j+1)}^{(k)} + \Delta z_{0,(j+1)}^{(k)}$ and $T_{(j+1)}^{(k+1)} = T_{(j+1)}^{(k)} + \Delta T_{(j+1)}^{(k)}$, where $k = 0,..,K_{(j+1)}$. The integer $K_{(j+1)}$ is the number of correction iterations required to converge on the solution and $k=0$ corresponds to the predictions $(z_{0,(j+1)}^{(0)}, T_{(j+1)}^{(0)})$. When one of the correction steps leads to a shooting function that converges, then the parameters $(z_{0,(j+1)}^{(k)}, T_{(j+1)}^{(k)})$ are stored and a new prediction is calculated as the algorithm advances to the $(j+2)^{th}$ step. In some cases the correction steps may diverge (i.e. the norm of consecutive shooting function values may increase instead of decreasing to zero), or they may converge very slowly. In order to address these issues, the shooting function values can be tracked to ensure that the norm of consecutive shooting functions is decreasing, and the number of correction steps can be bounded with $K_{\text{max}}$. In this work, when either of these constraints is not met, the step size is halved, $s(j) = s(j)/2$, Eqs. (95-97) are recalculated, and the correction procedure is repeated. Once the step size becomes small enough, the prediction usually becomes close enough to allow the shooting algorithm to converge.

The proposed algorithm can be used to calculate an entire branch of solutions. The starting and stopping frequencies must be input by the user. Additionally, the step size must be very carefully monitored and controlled (by maximum and minimum step size bounds and by setting the appropriate values of $K^*$). The algorithm typically converges and produces good results, but sometimes the algorithm parameters will need to be manually adjusted during the calculations to enable convergence or to avoid wasting computational effort with a sub-optimal step size.
### A2.6 Stability of Periodic Orbits

Using the proposed method, the stability of a periodic orbit can be calculated whenever convergence of a periodic solution is achieved (this was also discussed in [33]). The matrix \( \left[ \frac{\partial z(z_0, t = T, T)}{\partial z_0} \right] \) in Eq. (87) from the final Newton-Raphson calculation \((k=K(j+1))\) is the Monodromy matrix [40] of the periodic orbit. (Recall that it was formed from a linearization of the system about its periodic orbit and was calculated at \( t = T \).) The stability of this periodic orbit can be determined from the eigenvalues of this matrix. If the eigenvalues of the Monodromy matrix are inside (outside) the unit circle, then the orbit is stable (unstable). So the stability of periodic solution is conveniently found without additional calculations when using this method.

### A3 Appendix 3: Derivation of Nonlinear Stiffness of Beam Tip Spring

The nylon strip spring on the tip of the beam can be approximated as a tension on element, applying a force to the end of the beam along the line between the tip of the beam and the fixed support. When the beam tip is deflected by a small amount, \( \delta \), the nylon strip spring must deflect the same amount, so it forms a small angle, \( \theta \), with the right support. A diagram of this geometry is shown in Fig.60, where the nylon strip spring is assumed to be attached to the beam at the neutral axis. The nylon strip spring, which had original length \( l_s \), must stretch by an amount \( \Delta l_s \) (assuming the stretch in the beam is negligible). Treating the nylon strip spring as a massless rod in extension, the force is related to its displacement by, \( \Delta l_s = F l_s / (E_s A_s) \) where \( E_s \) is the elastic modulus of the nylon strip spring and \( A_s \) is the cross section area of the strip. Therefore, the tensile force is \( F = E_s A_s / l_s \* \Delta l_s \). A component of the tension force \( F \) is perpendicular to the beam and affects its bending motion, as shown in the diagram below.
The tip of the beam has a deflection of $\delta$, which can be used to write the strain in the nylon strip spring as a function of the undeformed length.

$$\Delta l_s = \sqrt{l_s^2 + \delta^2} - l_s$$  \hspace{1cm} (100)

The vertical contribution of the tensile force is $F_v = F \sin(\theta)$, or using the geometry of the figure and the previous relationship,

$$F_v = E_s A_s \left( \frac{\sqrt{l_s^2 + \delta^2} - l_s}{l_s + \sqrt{l_s^2 + \delta^2} - l_s} \right) = \frac{E_s A_s}{l_s} \left( \delta - l_s \delta \left( l_s^2 + \delta^2 \right)^{-1/2} \right)$$  \hspace{1cm} (101)

In order approximate the vertical force displacement relationship about small tip deflections of the beam, the previous equation was expanded in a Taylor series about $\delta = 0$ for terms to $O(\delta^3)$, and the higher order terms were neglected. This results in the following expression for the contribution of the nylon strip spring to the vertical force.

$$F_v \approx \frac{E_s A_s}{2l_s^3} \delta^3$$  \hspace{1cm} (102)

The nonlinear spring constant can therefore be approximated as $k_3 = E_s A_s / (2l_s^3)$. 

\[\text{Figure 60: Geometry for beam tip deflection and interaction of the strip of nylon strip spring}\]
References


