Eliminating Indefinite Mass Matrices with the Transmission Simulator Method of Substructuring

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ABSTRACT

The transmission simulator method of experimental dynamic substructuring captures the interface forces and motions through a fixture called a transmission simulator. The transmission simulator method avoids the need to measure connection point rotations and enriches the modal basis of the substructure model. The free modes of the experimental substructure mounted to the transmission simulator are measured. The finite element model of the transmission simulator is used to couple the experimental substructure to another substructure and to subtract the transmission simulator. However, in several cases the process of subtracting the transmission simulator has introduced an indefinite mass matrix for the experimental substructure. The authors previously developed metrics that could be used to identify which modes of the experimental model led to the indefinite mass matrix. A method is developed that utilizes those metrics with a sensitivity analysis to adjust the transmission simulator mass matrix so that the subtraction does not produce an indefinite mass matrix. A second method produces a positive definite mass matrix by adding a small amount of mass to the indefinite mass matrix. Both analytical and experimental examples are described.

1. Introduction

Experimental-analytical substructuring has been a topic of interest since modal testing was first introduced several decades ago. It is appealing because it has the potential to allow one to replace complicated subcomponents with experimental models that may be much less expensive to derive. It also allows the experimentalist to re-use the experimental model, predicting its response in a multitude of other configurations without repeating the test. One can also think of structural modification [1] as a special case of substructuring, where the modification is a special substructure that one wishes to determine in order to produce a desired response, (although the terms "substructuring" and "structural modification" are often used interchangeably [2]).

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The authors recently presented a new substructuring methodology, called Modal Constraints for Fixture and Subsystem (MCFS), that has proven quite effective at subtracting one structure from another [3]. Typically one has experimentally measured the modes of a built-up structure and one wishes to remove one subcomponent from that assembly. The subtraction is accomplished by creating a model of the subcomponent that is to be removed, making its mass, stiffness, and damping negative and then coupling the negative subcomponent to the assembly. Whereas, in conventional substructuring one enforces constraints between the points where the substructures are joined, the MCFS method estimates a set of modal coordinates on the substructure and enforces constraints on those coordinates. This reduces the sensitivity of the method to experimental errors and assures that an appropriate number of constraints is enforced.

The MCFS method is primarily used to estimate a modal model that can be used for substructuring predictions. The substructure is connected to a fixture or transmission simulator [4] and the assembly is tested in free-free conditions. This is equivalent to the well-known method where rigid masses are attached to the structure and used to create a mass-loaded interface, except that the proposed methodology is valid even if the transmission simulator is flexible. The transmission simulator serves to mass-load the interface of the subcomponent, enriching the modal basis and circumventing the need to measure displacements and rotations at the connection point. A model of the transmission simulator is then created and used to subtract its effects from the measured modal model in order to obtain a model for the substructure of interest in isolation, but with an improved modal basis. However, because a system with negative mass has been introduced in order to remove the transmission simulator, the substructure model may not necessarily have a positive definite mass matrix. Similar problems were encountered by other researchers when removing rigid masses from a structure [5]. This paper presents two methods that can be used to assure that the mass matrix of the subcomponent has positive mass.

2. Theory

2.1 Review of Subtraction of Modal Substructures

Suppose that the natural frequencies, ω_r, damping ratios, ζ_r, and matrix of mass-normalized mode shapes, $\mathbf{\Phi}_{\mathrm{C}}$, of an assembly consisting of the subcomponent of interest and the transmission simulator have been measured. The modal parameters of the transmission simulator are also known. (Here we shall refer to the substructure that is being removed as the transmission simulator, but in a general problem it could be any subcomponent that one wishes to subtract from the assembly). The assembly shall be referred to as system C and the transmission simulator as system A, as in [3], so the uncoupling procedure estimates the modes of B, the component of interest, since $\mathrm{C}-\mathrm{A}=(\mathrm{A}+\mathrm{B})-\mathrm{A}=\mathrm{B}$. First the equations of motion of C and (-A) are concatenated as follows

$$\begin{bmatrix} \mathbf{I}_{\mathrm{C}} & 0 \\ 0 & -\mathbf{I}_{\mathrm{A}} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{q}}_{\mathrm{C}} \\ \ddot{\mathbf{q}}_{\mathrm{A}} \end{Bmatrix} + \begin{bmatrix} \left[\diagdown 2\zeta_r\omega_r \diagdown \right]_{\mathrm{C}} & \mathbf{0} \\ \mathbf{0} & -\left[\diagdown 2\zeta_r\omega_r \diagdown \right]_{\mathrm{A}} \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{q}}_{\mathrm{C}} \\ \dot{\mathbf{q}}_{\mathrm{A}} \end{Bmatrix} + \begin{bmatrix} \left[\diagdown \omega_r^2 \diagdown \right]_{\mathrm{C}} & \mathbf{0} \\ \mathbf{0} & -\left[\diagdown \omega_r^2 \diagdown \right]_{\mathrm{A}} \end{bmatrix} \begin{Bmatrix} \mathbf{q}_{\mathrm{C}} \\ \mathbf{q}_{\mathrm{A}} \end{Bmatrix} = \begin{Bmatrix} \mathbf{\Phi}_{\mathrm{C}}^{\mathrm{T}}\mathbf{F}_{\mathrm{C}} \\ \mathbf{\Phi}_{\mathrm{A}}^{\mathrm{T}}\mathbf{F}_{\mathrm{A}} \end{Bmatrix}$$
$$\begin{Bmatrix} \mathbf{y}_{\mathrm{C}} \\ \mathbf{y}_{\mathrm{A}} \end{Bmatrix} = \begin{bmatrix} \mathbf{\Phi}_{\mathrm{C}} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Phi}_{\mathrm{A}} \end{bmatrix} \begin{Bmatrix} \mathbf{q}_{\mathrm{C}} \\ \mathbf{q}_{\mathrm{A}} \end{Bmatrix} \tag{1}$$

where the q dof are the generalized modal coordinates of each substructure, and then constraints are enforced as

$$\begin{bmatrix} \mathbf{\Phi}_{\mathrm{A},m}^{\dagger}\mathbf{\Phi}_{C,m} & -\mathbf{I}_{\mathrm{A}} \end{bmatrix} \begin{Bmatrix} \mathbf{q}_{\mathrm{C}} \\ \mathbf{q}_{\mathrm{A}} \end{Bmatrix} = 0 \tag{2}$$

Where the superscript, $\dagger$, denotes the pseudo-inverse of the matrix, and subscript *m* represents degrees of freedom common to both system C and system A that have been measured.

This is done by finding a matrix **B** that transforms the concatenated coordinates into a set of unconstrained coordinates. The coordinates of C are typically a suitable set [6], so one can choose

$$\begin{Bmatrix} \mathbf{q}_{\mathrm{C}} \\ \mathbf{q}_{\mathrm{A}} \end{Bmatrix} = \mathbf{B}\mathbf{q}_{\mathrm{C}}$$
$$\mathbf{B} = \begin{bmatrix} \mathbf{I}_{\mathrm{C}} \\ \boldsymbol{\tau} \end{bmatrix} \tag{3}$$

where $\boldsymbol{\tau} = \boldsymbol{\Phi}_{\mathrm{A},m}{}^{\dagger}\boldsymbol{\Phi}_{C,m}$. The number of modal coordinates in A and C are denoted N_A and N_C respectively. One can verify that **B** is in the null space of the matrix on the left in eqn. (2), so these coordinates always satisfy the constraints. As discussed in [3], if the model for the transmission simulator is accurate then the negative transmission simulator model completely cancels the forces that the transmission simulator would exert on system B. The equations of motion of the system after applying constraints are

$$\begin{gathered}\mathbf{M}_{\mathrm{B}}\ddot{\mathbf{q}}_{\mathrm{c}} + \mathbf{C}_{\mathrm{B}}\dot{\mathbf{q}}_{\mathrm{c}} + \mathbf{K}_{\mathrm{B}}\mathbf{q}_{\mathrm{c}} = \mathbf{B}^{\mathrm{T}}\begin{Bmatrix}\boldsymbol{\Phi}_{\mathrm{C}}{}^{\mathrm{T}}\mathbf{F}_{\mathrm{C}} \\ \boldsymbol{\Phi}_{\mathrm{A}}{}^{\mathrm{T}}\mathbf{F}_{\mathrm{A}}\end{Bmatrix} \\ \begin{Bmatrix}\mathbf{y}_{\mathrm{C}} \\ \mathbf{y}_{\mathrm{A}}\end{Bmatrix} = \begin{bmatrix}\boldsymbol{\Phi}_{\mathrm{C}} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Phi}_{\mathrm{A}}\end{bmatrix}\mathbf{B}\mathbf{q}_{\mathrm{c}}\end{gathered} \tag{4}$$

where the mass matrix of the constrained system is

$$\mathbf{M}_{\mathrm{B}} = \mathbf{B}^{\mathrm{T}}\begin{bmatrix}\mathbf{I}_{\mathrm{C}} & 0 \\ 0 & -\mathbf{I}_{\mathrm{A}}\end{bmatrix}\mathbf{B} = \mathbf{I}_{\mathrm{C}} - \boldsymbol{\tau}^{\mathrm{T}}\mathbf{I}_{\mathrm{A}}\boldsymbol{\tau} \tag{5}$$

and similarly for the damping and stiffness matrices.

The equation above shows that the eigenvalues of $\mathbf{M}_{\mathrm{B}}$ can become negative if any of the terms that are subtracted are too large. This might be the case, for example if the density value used in the transmission simulator model was higher than the true density. In that case one would simply need to adjust the modal mass of each the transmission simulator modes to alleviate the problem. In practice the situation is more complicated; negative mass is typically encountered due to a mismatch in the modal bases of A and C, causing the modal mass of A to be assigned incorrectly to the modes of C. Two approaches to adjust $\mathbf{M}_{\mathrm{B}}$ to make it positive definite are given in sections 2.2 and 2.3, the modal scale factor method and the added mass method. First the modal scale factor method is discussed.

2.2 Modal Scale Factor Method

The matrix $\boldsymbol{\tau}$ is related to an orthogonal projector that projects the motion of C onto the space spanned by the modes of the transmission simulator, A, as explained in [3]. One might be able to create a model for B that has a positive definite mass matrix by decreasing certain values along the diagonal in $\mathbf{I}_{\mathrm{A}}$ (or equivalently, by increasing the mode shape values on some of the modes of the transmission simulator by multiplying by a scale factor). With this approach, revise the name of the transmission simulator mass matrix from $\mathbf{I}_{\mathrm{A}}$ to $\mathbf{M}_{\mathrm{A}}$. $\mathbf{M}_{\mathrm{A}}$ begins as the identity matrix, but individual elements will be reduced as described below. The eigenvalues of the mass matrix, $\mathbf{M}_{\mathrm{B}}$, are found by solving the eigenvalue problem,

$$\mathbf{M}_{\mathrm{B}}\boldsymbol{\psi}_k = \lambda_k\boldsymbol{\psi}_k \tag{6}$$

There are typically only a few negative eigenvalues $\lambda_k < 0 \quad k = 0 \ldots N_{\mathrm{neg}}$ and several modal masses, so this can be cast as an under-constrained optimization problem where one seeks values for the modal masses of the transmission simulator, $M_{TS,j}$, in $\mathbf{M}_A = diag\lfloor M_{TS,1} \quad \ldots \quad M_{TS,N_A} \rfloor$ that cause all of the eigenvalues of $\mathbf{M}_{\mathrm{B}}$ to be positive. An equivalent approach is to reduce the diagonals of $\mathbf{M}_{\mathrm{A}}$ until all of the eigenvalues of $\boldsymbol{\tau}^T\mathbf{M}_{\mathrm{A}}\boldsymbol{\tau}$ are ***less*** than one (see eqn. (5)). A simplified Newton-Raphson algorithm can be devised to accomplish this. First calculate the Jacobian (sensitivity matrix) of the eigenvalues which are greater than one in $\boldsymbol{\tau}^T\mathbf{M}_{\mathrm{A}}\boldsymbol{\tau}$ with respect to each diagonal member of $\mathbf{M}_{\mathrm{A}}$. Let $\mathbf{f}(M_{TS,j})=[\lambda_{\mathrm{k}} \ldots \lambda_{\mathrm{N>1}}]^T$. A first order Taylor series expansion of this function is

$$\mathbf{f}_{desired}(M_{TS,j}) = \mathbf{f}(M_{TS,j}) + [\nabla\mathbf{f}(M_{TS,j})]_{M_{TS,j}}\boldsymbol{\Delta}M_{TS,j} \tag{7}$$

where $\mathbf{f}_{desired}(M_{TS,j})$ is chosen as a new eigenvalue slightly below one (.9999) for this work. Instead of proceeding in the usual way with the full matrices, one chooses the worst (largest) eigenvalue of $\boldsymbol{\tau}^T\mathbf{M}_{\mathrm{A}}\boldsymbol{\tau}$ and only the mass value of $\mathbf{M}_{\mathrm{A}}$ which is most sensitive to the worst eigenvalue. This reduces the matrices down to scalars, which alleviates some problems if the matrices have high condition numbers. Then one solves for the reduction in a single diagonal member of $\mathbf{M}_{\mathrm{A}}$ as

$$\boldsymbol{\Delta}M_{TS,j} = [\nabla\mathbf{f}(M_{TS,j})]^{-1}{}_{M_{TS,j}}[\mathbf{f}_{desired}(M_{TS,j}) - \mathbf{f}(M_{TS,j})] \tag{8}$$

where there is only one member in the Jacobian and one eigenvalue to change. The eigenvalues are a nonlinear function, so the process is iterative. On each iteration, the mass that is most sensitive to the largest eigenvalue above one is adjusted, until all eigenvalues of $\boldsymbol{\tau}^T \mathbf{M}_A \boldsymbol{\tau}$ are below one. Then $\mathbf{M}_B$ will have no negative eigenvalues.

The modal scale factors found using this approach will be optimum only in the sense that they are the smallest perturbations to the transmission simulator mass matrix that produce a positive definite $\mathbf{M}_B$.

2.3 Added Mass Method

The procedure in Section 2.2 seeks to find a positive definite mass matrix by adjusting the modal scale factors (or the modal mass) of the transmission simulator. An alternative is to simply compute $\mathbf{M}_B$ in eqn. (5) and then to add mass to the system to cause the negative eigenvalues of the mass matrix to become slightly greater than zero. Physically, this is identical to attaching point masses to the system at various locations, which are governed by equations of motion $\mathbf{m}\ddot{\mathbf{y}}_C = 0$, where $\mathbf{m}$ is a diagonal matrix of point masses. One can use the same procedure described in eqns. (1) through (4) to compute the equations of motion for the system with these additional masses, and the resulting equations are identical to those in eqn. (4) only with $\mathbf{M}_B$ replaced with $\mathbf{M}_B + \Delta\mathbf{m}$, where $\Delta\mathbf{m} = \boldsymbol{\Phi}_C{}^T \mathbf{m} \boldsymbol{\Phi}_C$.

Typically there are more physical nodes than there are modes, so $\boldsymbol{\Phi}_C$ has more rows than columns and one can determine a pattern for the applied masses that would create an arbitrary increase, $\Delta\mathbf{m}$, in the mass of the system. The smallest change to $\mathbf{M}_B$ will cause the negative eigenvalues to increase to just above zero while leaving the remaining eigenvalues unchanged, and can be obtained by adding the following to each eigenvalue.

$$\Delta\lambda_k = \begin{cases} 0 & \lambda_k > 0 \\ -\lambda_k + \varepsilon & \lambda_k \le 0 \end{cases} \tag{9}$$

One can then find $\Delta\mathbf{m}$ using,

$$\Delta\mathbf{m} = \boldsymbol{\psi}\hat{\boldsymbol{\Lambda}}\boldsymbol{\psi}^T \tag{10}$$

where $\hat{\boldsymbol{\Lambda}} = diag\left[\left(\lambda_1 + \Delta\lambda_1\right) \quad \cdots \quad \left(\lambda_{Nc} + \Delta\lambda_{N_C}\right)\right]$ and $\boldsymbol{\psi} = \left[\boldsymbol{\psi}_k \quad \cdots \quad \boldsymbol{\psi}_{N_C}\right]$. The eigenvalues of $\mathbf{M}_B + \Delta\mathbf{m}$ will then be the strictly positive values that are on the diagonal of $\hat{\boldsymbol{\Lambda}}$. One would hope that very little mass would need to be added to the structure to make the eigenvalues positive. One can measure the amount of mass added using the ratio of the norms: $n_{rat} = \frac{\|\Delta\mathbf{m}\|}{\|\mathbf{M}_B\|}$.

3. Applications

3.1 Analytical T-Beam System

The first system considered is a 12-inch long steel beam with a 0.75 by 1.0 inch cross section. This is the same system that was considered in [6]. A 6.0-inch long transmission simulator with the same cross section is attached to the beam and the modes of the assembly (the C system) are computed; in the usual practice these modes would be found experimentally with a modal test, but here the analytically computed modes are used in order to simulate perfect measurements.

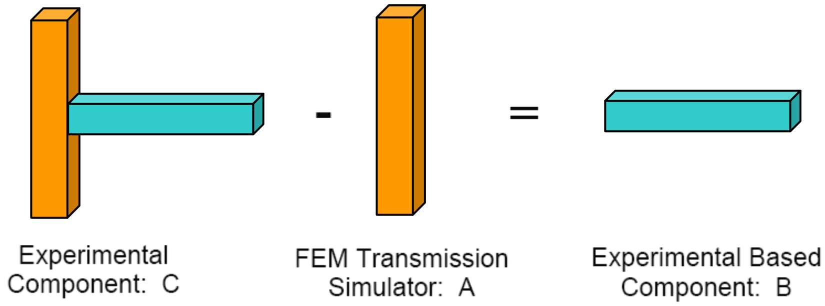

Figure 1: T-Beam System. The substructure of interest is the (blue) horizontal beam. It is tested while connected to the (orange) vertical beam, or transmission simulator. The proposed procedures will be used to obtain a positive definite mass matrix after removing the transmission simulator.

As discussed in [6], it is important to assure that enough modes of the transmission simulator are used to span the space of the motions of C in the frequency band of interest. Using the best practice identified in that work, seven modes are used for the transmission simulator and 15 modes for C, which corresponds to all modes below 20 kHz for C and below 24 kHz for A. The resulting model for B has two negative eigenvalues: λ_1= -0.00050468, λ_2=-2.4058e-16, although λ_2 is essentially zero. We desire to make these eigenvalues positive using the methods presented in Section 2.

3.1.1 Mode Scale Factor Method

The mode scale method was applied to this system using the scalar Newton-Raphson algorithm. The Jacobian was used to identify the two modes of the transmission simulator that most strongly influenced the negative eigenvalues, revealing that the first and sixth modes had the largest influence. These were the same modes that were identified by the metrics in [6] as having the largest contribution to the negative eigenvalues. The Newton-Raphson algorithm converged to values of $M_{TS,1}$=0.9984 and $M_{TS,6}$=0.9999, which correspond to multiplying the first and sixth mode shapes of the transmission simulator by 1.0008and 1.000050 respectively. The smallest two eigenvalues of $\mathbf{M}_{\mathrm{B}}$ were positive values on the order of 1e-5.

It is important to mention that some difficulty was encountered in selecting a value of $\mathbf{f}_{desired}(M_{TS,j})$ for these calculations. Initially $\mathbf{f}_{desired}(M_{TS,j})$ =0.99 was used in which case the Newton-Raphson algorithm gave results in which the FRFs had extra zeros in the response. When more significant figures were used (0.9999) the results converged to analytical up to 20 kHz.

3.1.2 Added Mass Method

The added mass method was also applied to this system. An addition to the mass matrix, $\Delta\mathbf{m}$, was computed using eqns. (9) and (10) with ε=2.2e-14 (one hundred times larger than the estimated machine precision) and added to $\mathbf{M}_{\mathrm{B}}$. As expected, the smallest eigenvalues of $\mathbf{M}_{\mathrm{B}}$ became 2.2e-14. The amount of mass that has been added can be quantified using the ratio between the norm of the added mass and the total mass of $\mathbf{M}_{\mathrm{B}}$, which was found to be $n_{\mathrm{rat}} = 0.000505$, indicating that a minuscule amount of mass has been added to the system in order to make $\mathbf{M}_{\mathrm{B}}$ positive definite.

Now that the model has positive mass, it is important to check that it still represents the system accurately. In previous works the authors have found that very small changes to the modal scale factors can sometimes introduce errors causing significant changes to the model. These are easiest to assess by reconstructing the frequency responses (FRFs) of the model, since they give a visual indication of whether both the frequency and amplitude of each mode is correct. Figures 2 and 3 show the FRFs of the system that were obtained after using each of the proposed methods to correct for the negative eigenvalues in the mass matrix.

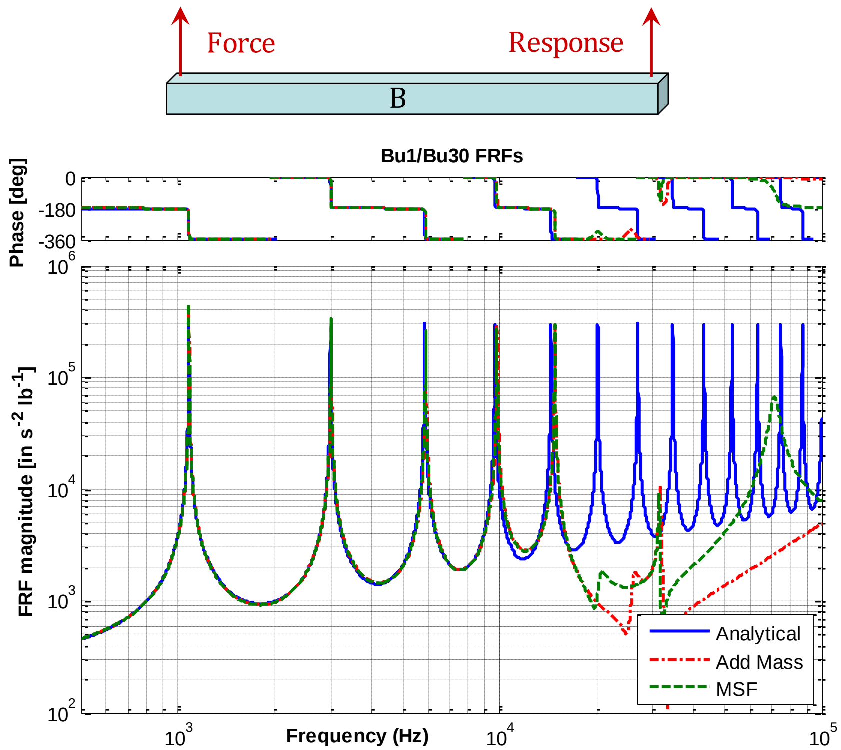

Figure 2: Frequency response function of beam in bending direction after removing the transmission simulator. The blue line shows the analytical FRF computed from the FEA model of the beam alone, while the red and green lines show those of the substructure model estimated after removing the transmission simulator and after adjusting the mass matrix using the Added Mass and Modal Scale Factor approaches.

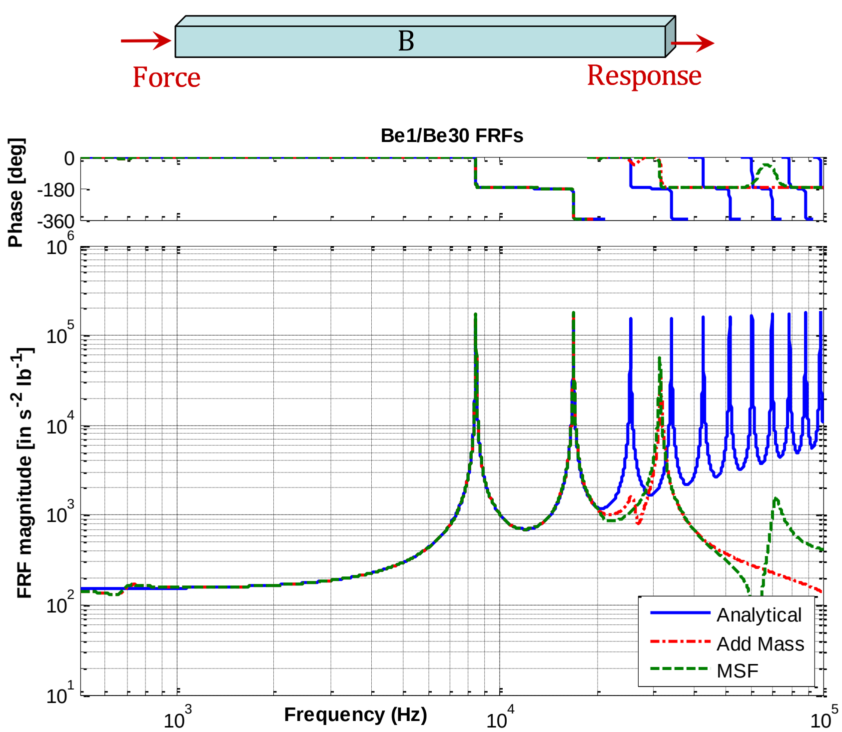

Figure 3: Frequency response function in the axial direction. (See description for Fig. 2)

Figures 2 and 3 reveal that the substructure models accurately reproduce all of the modes of the system out to almost 20 kHz, which is not too surprising since the set of modes used to derive them spanned this same range. (According to the traditional rule of thumb, one would only expect the model to be accurate out to about half the bandwidth of the modes used to derive it.) Both the Added Mass and Modal Scale Factor approaches produce nearly identical results in this bandwidth, accurately reproducing the first five bending and two axial modes. This is significant, since, as described in [6], the authors previously tried many different adjustments to the substructure model and were unable to obtain a model that reproduced the FRFs accurately. For example, Section 5.1 in the Appendix shows the result obtained when using an ad-hoc approach to eliminate the negative mass, where there are fairly significant errors in the FRFs predicted by the substructure model.

3.2 Three Dimensional Beam-Plate System

In this section the proposed methods are applied to the system studied in [3], a schematic of which is repeated in Fig. 4. The experimental subcomponent, C, consists of a beam connected to a plate at its center. A transmission simulator, A, is attached and a modal test is performed, as shown in the photograph. System C is attached to the transmission simulator with eight bolts around the circumference of the plate. The finite element model consists of the cylinder plus a second copy of the transmission simulator. Modal substructuring is used to assemble C and D and to remove two copies of the transmission simulator, resulting in the built-up structure, E.

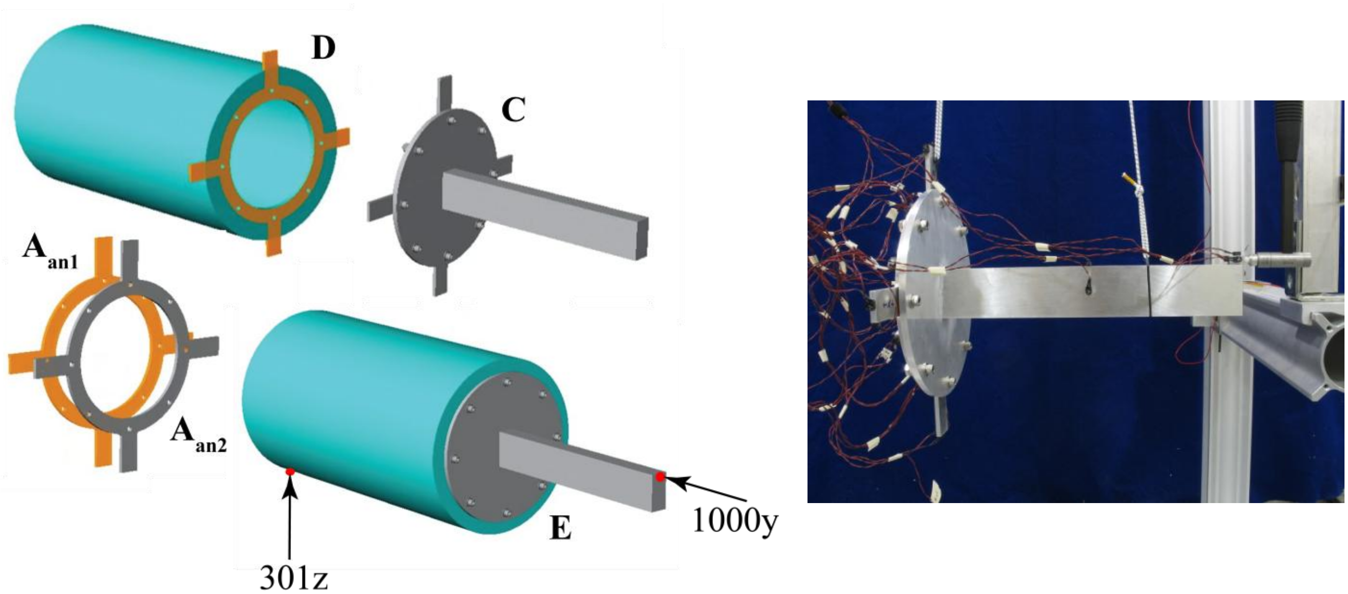

Figure 4: Schematic of Cylinder-Plate System and photograph of the hardware that was tested to obtain an experimental model for substructure C. Labels are shown indicating the names of the substructures, as well as two drive points where FRFs were later reconstructed.

As explained in [3], two copies of the transmission simulator were used in this problem to simplify the coupling between D and C. Using this approach, the same modal constraints that are used to connect C or D to the negative transmission simulator model are also used to connect C to D. Since the transmission simulator is welded to D, this results in an effective, continuous constraint between the two systems. Furthermore, because the bolted joints are part of system C, their effect is captured experimentally as it is manifest in the coupling between C and the transmission simulator.

Because there are two transmission simulator models in this problem, the methods proposed in this work must be implemented differently. The C system could be treated in the usual way, but one must also treat subcomponent D as if it were an experimental subcomponent (although with a large number of perfectly measured modes, since they can be readily extracted from the FEA model). Then one could remove A from D, assure that the mass is positive and return the resulting model to the FEA package. On the other hand, either of the methods proposed here could potentially be implemented within FEA, eliminating the negative mass after the subcomponents had been assembled but before computing the modes or response of the assembly. Both of these approaches will be explored briefly.

The experimental test identified 25 modes of the C system, the highest having a natural frequency of 3835 Hz. Eighteen modes were used to model the transmission simulator spanning 0 to 1850 Hz. To facilitate the implementation, the D system was reduced to a 100-mode model spanning 0 to 6165 Hz. When all of these components are assembled, the resulting system has four negative eigenvalues: $\lambda_1= -0.197$, $\lambda_2= -0.0764$, $\lambda_3= -0.134$, $\lambda_4= -0.118$. The negative mass causes the system to have four spurious natural frequencies that are purely imaginary f_1=2144i, f_2=2636i, f_3=2813.3i, f_4=3453.3i Hz. Plots of the reconstructed FRFs are shown in [3], revealing that these spurious natural frequencies apparently do not contaminate the FRFs noticeably, but a positive definite mass matrix is still desired, so the methods presented here will be employed.

3.2.1 Case 1: Assemble all substructures then correct the mass matrix

First consider the case where all of the substructures are assembled before correcting the mass matrix. (In the applications of interest, this would require one to embed the mass correction algorithm within the FEA code.) The added mass method was used to compute a matrix $\Delta\mathbf{m}$ that would make the mass matrix positive definite. The resulting mass addition was relatively large compared to the original mass matrix, with a norm ratio of $n_{\mathrm{rat}} = 0.197$. The frequency response functions were reconstructed at two points in order to determine whether the added mass had changed the nature of the response. Figures 5 and 6 show the frequency responses before and after applying the mass addition algorithm, as well as the finite element truth model described in [3].

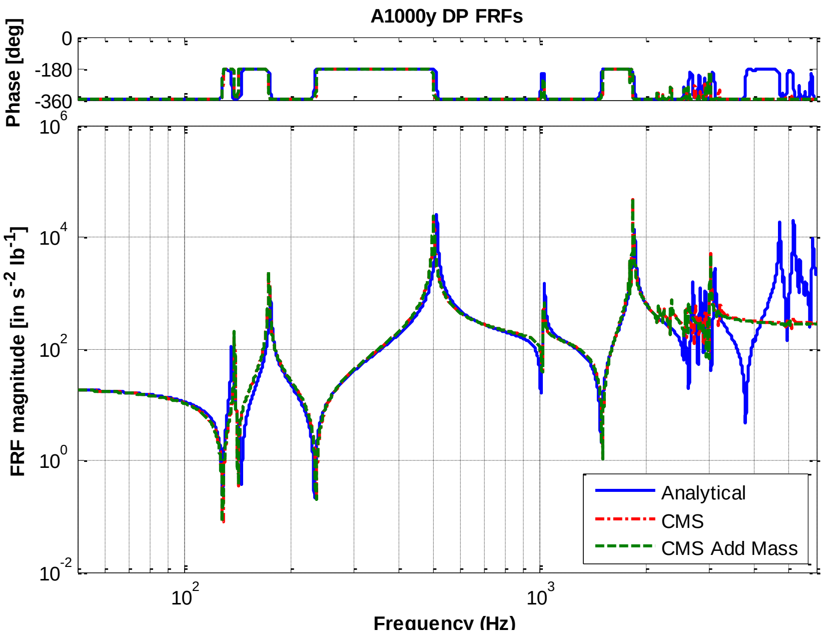

Figure 5: Frequency response function of the assembled system at the drive point labeled 1000y (axial direction). The blue line shows the analytical FRF of a finite element truth model. The other lines show the FRFs estimated by modal substructuring, the red is the baseline model that has negative mass while the green is from a model that was corrected using the added mass algorithm in Sec. 3.1.2.

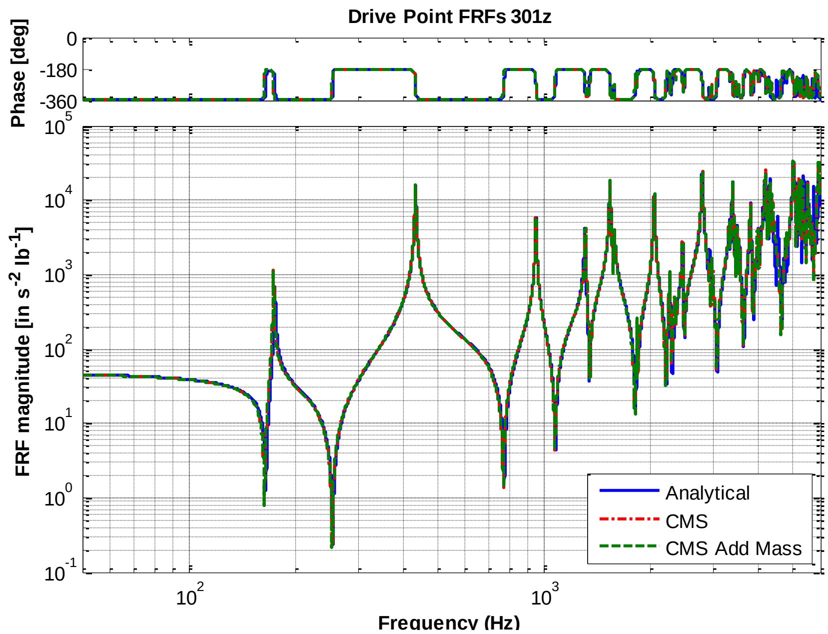

Figure 6: Frequency response function of the assembled system at the drive point labeled 301z (radial direction). (See description for Fig. 5)

3.2.2 Case 2: Correct the mass matrix for C-A

Next consider the case where we wish to create a model for subcomponent C that has positive definite mass after removing the transmission simulator A. Coupling C-A directly one finds that the resulting system's mass matrix has two negative eigenvalues: λ_1=-0.116 and λ_2=-0.0865. There are eight imaginary natural frequencies, although three of them are above 100 kHz and three are below 1 Hz. The other two are at 0+2304.4i and 0+2532.1i Hz, within the range of the modes of C.

First the added mass method was used to correct the mass matrix for this system. It computed a mass addition whose norm was $n_{\text{rat}} = 0.116$. The resulting system no longer had the two imaginary natural frequencies in the band of interest.

Next the mode scale factor method was employed. This algorithm reduced five of the 18 modal masses of the transmission simulator as follows: mass 1 = .9931; mass 9 = .5783; mass10=.9488; mass 11=.8224; and mass 13=.4942. This achieved a positive definite mass matrix. The resulting axial FRF of the axial response of the plate and beam is shown in Figure 7. The mode scale method has slightly more degradation than the mass added method as compared with the analytical result.

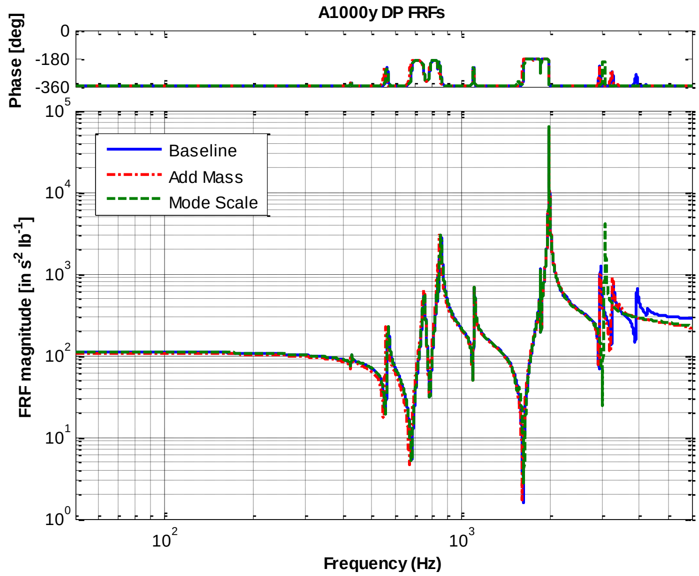

Figure 7: Blue curve provides the analytical axial C-A plate/beam FRF, red is the added mass method and green is the mode scale factor method for generating positive definite mass matrix after subtraction

3.3 Observations on the transmission simulator mass corrections based on the applications

In the T beam studies, only a small amount of mass had to be subtracted from the transmission simulator using the modal scale factor method, and only a small amount had to be added with the added mass approach. With the cylinder/plate/beam system, the amount of mass subtracted by the modal scale factor method or added by the added mass method was much larger. It was also observed that the FRF quality of the final result for the modal scale factor method degraded when larger mass changes were required to make the constrained mass matrix positive definite. The authors believe that the severity of the correction is related to adequacy with which the mode shapes of the transmission simulator span the space of the actual motion of the substructures when they are constrained together. The transmission simulator mode shapes probably never span the constrained connection motion perfectly. If the transmission simulator mode shapes span the space well, only small corrections are needed, but larger corrections are required when they do not span the space as well. Therefore the amount of mass change required may provide a metric on the quality of the transmission simulator mode shapes for use in the specific application.

4. Conclusions

This work presented two methods that can be used to obtain a positive definite mass matrix from the indefinite result that is often obtained when removing one structure from another using modal substructuring. The first method, called the mode scale factor method, used a nonlinear optimization algorithm to vary the modal masses (or equivalently mode scale factors) of the modes of the transmission simulator. This approach was found to work well in cases where the required transmission simulator modal mass changes were small, and it has the advantage that it is physically justifiable whenever the mode scale factors found are reasonable considering the uncertainty in the transmission simulator model. When large changes in the mode scale factors were required, the resulting response FRFs degraded slightly in the applications investigated here.

The second method corrects the system of interest by adding mass to the structure. A simple equation is available to compute the amount of mass required to make the mass matrix positive definite. This approach appears to be more robust than the modal scale factor method. However, one disadvantage of this approach is that one cannot readily relate the mass added to the transmission simulator modes, which are the cause of the negative mass in the first place.

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5. Appendix

5.1 Results using ad-hoc approach to eliminate negative mass for T-Beam system

The following figures show the FRFs that were obtained using an ad-hoc approach to eliminate the negative mass. The modal mass of all of the transmission simulator modes was decreased (simulating a uniform decrease in density of that model) until the negative eigenvalues disappeared. Trial and error revealed that the modal scale factors had to be multiplied by 1.011 to eliminate all of the negative eigenvalues.

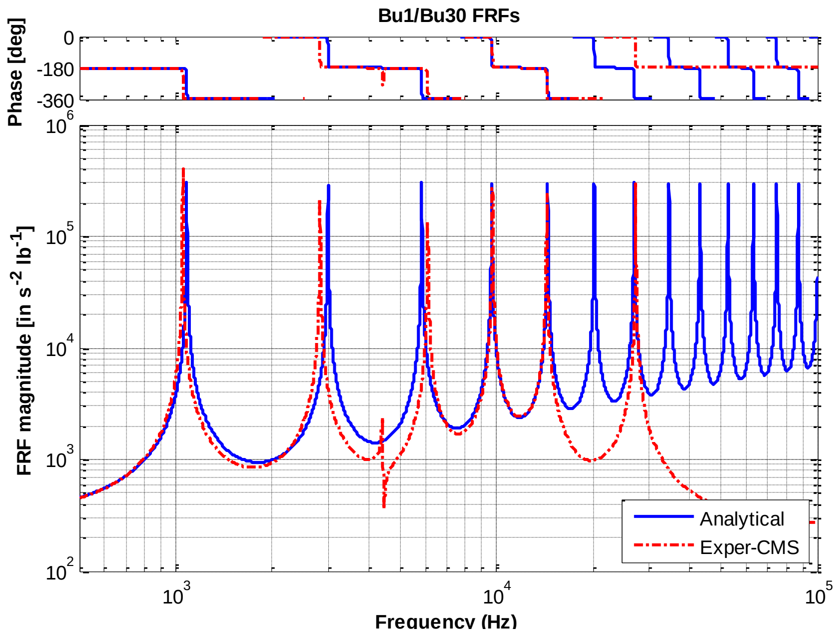

Figure 8: Frequency response function of beam in bending direction after removing the transmission simulator. The blue line shows the analytical FRF computed from the FEA model of the beam alone, while the red line shows that of the substructure model after eliminating the negative eigenvalues using the ad-hoc approach.

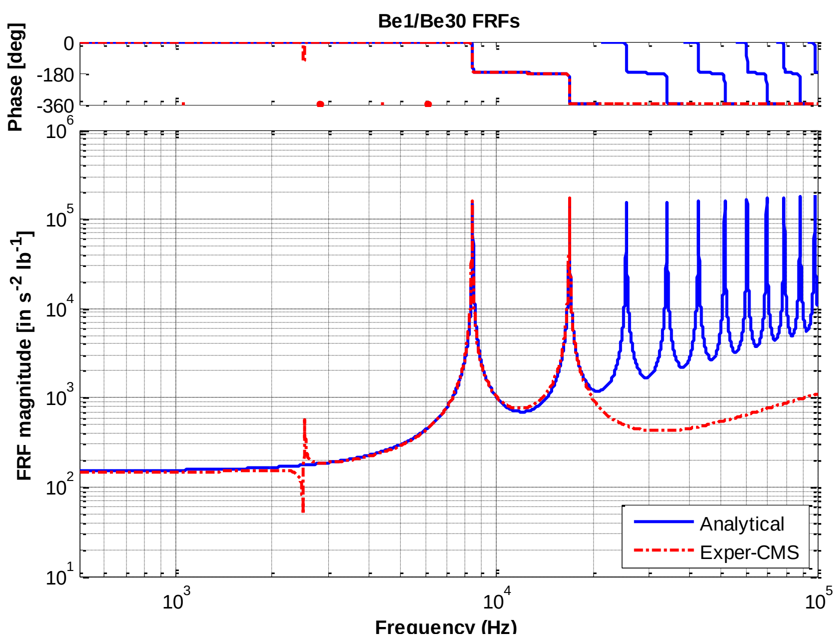

Figure 9: Frequency response function in the axial direction. (See description for Fig. 8)