Geometrical Theory of Nonlinear Modal Analysis

by

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Abstract

This thesis presents a geometrical theory for modal analysis of nonlinear structural systems. The first objective of this thesis is to develop a coherent and consistent framework, wherein, dynamic analysis of a class of nonlinear systems of large order can be performed efficiently. Specifically, methods are presented to compute the periodic responses or nonlinear modes of the system, to assess their stability, and to perform bifurcation analysis in order to characterize the branches of solutions that emerge.

In this study, the definition of nonlinear eigensolutions (modes) is based on the concept of an Instantaneous Center Manifold (ICM) which was introduced by the author as the periodic quotient of the invariant manifolds that can be defined for a class of nonlinear systems. Both analytical and numerical methods for calculation of such invariant manifolds have been developed. Also an efficient numerical method for calculation of nonlinear modes, namely Multi-harmonic Multiple-point Collocation (MMC), has been developed which can identify multiple nonlinear modes in each solution and which does not require integrating the equations of motion.

The second objective is to study extensions to superposition for nonlinear systems, where the response of the system can be expressed as a function of its nonlinear eigensolutions (modes). More specificity, it is of interest to find the general form of these functions, which are called connecting functions and can be used to generate new solutions to the system from combinations of the nonlinear modes. The form of the connecting function for a class of nonlinear systems is presented and methods are presented for computing them numerically. This work could eventually allow one to obtain an arbitrary solution of the system from a set of its eigensolutions, namely nonlinear modes. Second, they can be used to decompose any arbitrary solution of the system onto a set of its neighboring eigensolutions in order to better understand the system characteristics that cause the response. Three numerical approaches have been also developed to identify connecting functions which provide interesting insights into the relationship between a system’s eigensolutions and its general solution.
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Chapter 1

Research Scope and Contributions

This thesis presents a geometrical theory for modal analysis of nonlinear structural systems. More precisely, we primarily consider nonlinear conservative systems which can be described by a set of second order, ordinary differential equations with at least one marginally stable equilibrium point. The proposed geometrical theory has led, first and foremost, to a better understanding of the dynamics of nonlinear structural systems and second, to aid in developing efficient numerical modal analysis techniques of dynamic analysis of large engineering structures with strong nonlinearities. In the future this could have applications in order reduction and sub-structuring in the future.

1.1 Background

Modal analysis is perhaps the most popular approach for analysis of linear systems in structural dynamics [50, 69, 94]. Its wide acceptance stems from its generality since the an arbitrary response of a linear structural system can be easily generated from eigensolutions of the system, i.e. its linear modes. This property is the cornerstone of all modal analysis techniques such as order reduction and sub-structuring, which are essential when dealing with large structures. However, most structural systems, at least at some energy levels, exhibit nonlinearities which can not be captured by linear analysis techniques. In
this regard, many have attempted to extend some of these techniques to nonlinear systems. Although these attempts led to significant contributions and paved the way for further advancements, to the best knowledge of the author, they have not addressed the main challenge in dynamic analysis of nonlinear structures, namely, finding a general solution for the response of a nonlinear structural system from its eigensolutions.

This thesis uses a geometrical approach to develop a theory for modal analysis of nonlinear structural systems. This approach will be used to try to overcome the two main obstacles that prevent us from finding a general solution for a nonlinear system, namely, linear independence and superposition. In order to explain these obstacles, we need to adopt a more precise language and revisit linear modal analysis.

In this regard, suppose that
\[ \dot{z} = Az, \; z \in B^{2n} \] (1.1)
represents a \( n \) degree-of-freedom (DOF) linear structural system where \( B \) is the Banach space and also suppose that \( S \subseteq B^{2n} \) is the set of all solutions of the system. Thus, one can define a connecting function of the system in Eq.(1.1) as a (vector) function \( F : S \rightarrow S \) that connects a set of the solutions of the system to another solution of the same system. In other words,
\[ F(\alpha_1 x_1, \ldots, \alpha_n x_n | \forall x_i \in S \land \forall \alpha_i \in \mathbb{R}) = x_{n+1} \in S. \] (1.2)

Using this definition, superposition simply points out that all connecting functions of all the solutions of the system (1.1) have to satisfy both additivity and homogeneity [21]. In other words, for a linear system
\[ F(\alpha_1 x_1, \ldots, \alpha_n x_n | \forall x_i \in S) = \alpha_1 x_1 + \cdots + \alpha_n x_n. \] (1.3)
In this sense, one can show that the general solution of the system exists by simply proving that $x_{n+1}$ covers the entire set $S$ or equivalently the connecting function $F$ is a bijective [129, 32].

Moreover, if one wishes to confine the set of solutions $x_i$ to be a finite subset of $S$, i.e. to find a surjective $F$, then the problem changes into finding a finite number of linearly independent solutions $x_i$. Finding $n$ linearly independent solutions of (1.1) at once leads to an indeterminate system of algebraic equations which can not typically be solved. To overcome this problem, it is represented as an eigenvalue problem where the mentioned algebraic system can be solved in $n$ consecutive steps\footnote{Note that eigensolutions are a small subset of all linearly independent solutions.} [72, 21]. Therefore one can conclude that for any solution $x_{n+1} \in S$, the connecting function $F$ can be uniquely found if (only) $n$ linearly independent solutions of the system (instead of all the solutions in $S$) are known. Conversely, because of superposition, all solutions in $S$ can be constructed using only $n$ linearly independent solutions of the system. Therefore the concepts of linear independence and superposition play crucial and complementary roles in the processes of breaking the space of all solutions of a linear system down into independent elements and reconstructing solutions from those elements. Loop 1 in Figure 1.1 illustrates this connection.

Figure 1.1: Linear modal analysis in a glance. LI stands for linearly independent.

Linear independence and superposition are critical to linear modal analysis, and hence it is important
to seek an extension to nonlinear systems. For instance, if the eigensolutions of a linear system were not linearly independent, then it would not be possible to decouple or to reduce the order of the system [50, 38].

Therefore, in order to extend the same methodology to nonlinear systems, one needs to first define eigensolutions of nonlinear systems based on an extension of the concept of linear independence and second find the general form for the connecting functions of a nonlinear structural system.

1.2 Overview of the Document

The rest of this document is organized as follows. Chapter 2 briefly reviews different definitions of nonlinear modes. Moreover, it explains how the concept of invariance can be used as an alternative to linear independence in a nonlinear system to define its eigensolutions. The ICM concept and a definition of nonlinear modes based on the ICM is also presented. In Chapter 3, numerical and analytical methods are presented for finding nonlinear modes. Stability analysis of nonlinear modes is addressed in Chapter 4. Bifurcation of nonlinear modes is also studied in this chapter. Moreover, the orientability of local ICMs and its connection with bifurcation of nonlinear modes is explained. Connecting functions of nonlinear systems are investigated in Chapter 5. This chapter also includes three numerical approaches to identify connecting functions as solutions of the system. Chapter 6 concludes this document with a summary of the main contributions and an outlook of the potential future works.

The studies presented in this document are extracted from published, submitted or soon to be submitted papers by the authors in the following order. Chapter 2 is based on [5] and a completed study which will appear in [6]. Methods presented in Chapter 3 are extracted from the published and under-review papers [5] and [8]. Chapter 4 is mainly based on the study which was published in part in [7]. The rest of the study will appear in [9]. And finally materials presented in Chapter 5 outline the

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2Each chapter includes a review of related research.
studies done in [12] and a forthcoming paper [13], respectively.
Chapter 2

Nonlinear Modes of Vibration

The concept of linear modes is defined rigorously for linear systems and implemented in analysis of free and forced oscillatory systems in linear vibration theory [136, 110]. Linear modes, as eigensolutions of a linear system of (homogenous) second order differential equations have two essential properties. First, they are periodic \(^1\) and second, they are linearly independent \(^2\). Because of the latter property, if a motion is initiated on one, and only one linear mode, it does not excite the remaining modes. These two properties provided the basic foundation of all efforts to extend the concept of linear mode to nonlinear systems.

The idea of extending the definition of linear mode to nonlinear systems is not new. Since the late 1960’s, many have attempted to characterize a set of solutions of a nonlinear system as nonlinear modes which led to various definitions for nonlinear modes. Perhaps, the most noticeable definitions are the ones proposed by Rosenberg [117, 114, 118, 115, 116], Shaw and Pierre [28, 74, 122, 123, 124] and Vakakis, Kerschen and others [80, 81]. We summarize these definitions briefly here and later represent them in detail in Section 2.

The existence, properties and approximation of vibration modes of nonlinear systems were presented

\(^1\)In non-conservative cases, state space modes are considered as LNMs.

\(^2\)For linear systems, orthogonality of linear modess can be readily obtained from this property.
first in the seminal works by Rosenberg [114, 117, 116, 118, 115]. In direct analogy to the linear modes, vibration modes of conservative nonlinear systems were defined as periodic motions of such a nature that all masses pass through their equilibrium positions simultaneously [114]. This definition was later simplified to vibration in unison of the system, i.e. a synchronous oscillation, by Vakakis [137]. Vibration modes of nonlinear systems, which were initially introduced as the natural modes of nonlinear systems [117], renamed to Nonlinear Normal Modes (NNM) by Rosenberg merely because the equivalent phenomenon in linear systems was vibration in normal modes [114]. However, no normalization process was defined in the proposed definition. Study of NNMs was further pursued in the works of Rand [107, 108, 109], Greenberg and Yang [60] and Yen [142].

Four decades later, Rosenberg’s definition was modified by Kerschen and others [80, 81] by relaxing some constraining conditions in order to allow the definition to accept all periodic solutions as NNMs. For instance, the modified definition allowed cases of internal resonance, where coordinates may not oscillate synchronously, to be considered as NNMs. However, in this definition, finding a modal relation, i.e. the mode shapes, was no longer associated with finding the NNMs of the system and NNMs were defined only based on their periodic characteristic such as their (fundamental) frequency.

Nearly all the previous work on NNMs dealt exclusively with conservative systems until the efforts of Shaw and Pierre [123, 122, 124] to extend the concept of NNMs to a class of general \( n \)-DOF oscillatory systems which can include non-conservative systems. Shaw and Pierre defined a NNM as a motion that takes place on a two-dimensional invariant manifold which is tangent to the vector field at a stable equilibrium [122, 123]. In other words, these invariant manifolds were defined as the loci of NNMs of the system in the state space. In general, these invariant manifolds are nonlinear surfaces but in the case of linear systems they represent planes which contain all the LNMs.

Although, these definitions paved the way to define and compute a special set of solutions of nonlinear

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3Note that NNMs, unlike linear normal modes, are not mutually orthogonal.
systems, there are still some limitations such as questions and phenomena that can not be answered or explained using these definitions. Some of these limitations are explained next.

For $n$-DOF Hamiltonian systems there exist basic theorems by Themoa [93] and Weinstein [138], stating that for each fixed level of energy there exist at least $n$ periodic solutions (normal modes) passing through each stable equilibrium. This explains the phenomenon, as discussed in [116], that the number of nonlinear normal modes may exceed the number of DOF of the system. Moreover, some of those solutions may not be analytic continuations of linear modes [116, 137] and therefore can not be found using continuation as utilized by Peetes et al [105] to compute NNMs (more details are provided in Section 2).

Furthermore, the concept of invariance can be viewed as an extension of the concept of linear independence for nonlinear systems, since, solutions initiated on an invariant manifold stay on it for all time and do not pass through other manifolds. The latter property is true, because otherwise, there would exist at least two intersecting manifolds with non-unique tangent planes at their intersection which contradicts the definition of manifold (see [130, 14, 91]).

Therefore, one may assume that each one of aforementioned approaches, i.e. Vakakis’s modification to Rosenberg’s definition and Shaw and Pierre’s, extended the concept of linear modes to nonlinear systems by extending only one essential property of linear modes while leaving the other out. In addition, considering the fact that invariance of a manifold, in general, does not imply periodicity of the solutions on it and vice versa, one may wonder weather any of the solutions on the invariant manifolds are periodic? Does the space of periodic solutions of (2.1), i.e. NNMs based on the modified Rosenberg definition, overlap with the space of invariant solutions, i.e. NNMs based on Shaw and Pierre’s invariant manifold definition? and finally, do the solutions, that are not analytic continuations of linear modes, appear on any invariant manifold?

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4This is a more general condition than being conservative, since while including all conservative systems, Hamiltonian systems may not be conservative.
Moreover, the outlined definitions carry some intrinsic limitations. For instance, Rosenberg’s and Vakakis’ approaches can’t be applied to non-conservative systems (see Section 2.1 for detail). On the other hand, Shaw and Pierre’s invariant manifold approach also has certain limitations. Its accuracy highly depends on the order of the approximation of the invariant manifolds and the process is computationally expensive. This drawback becomes more evident, when Shaw and Pierre’s approach is used to simplify the nonlinear system by transforming the governing differential equations of motion to a new set of differential equations governing the motions on the invariant manifolds. Any imperfection in calculating the invariant manifolds will directly affect the accuracy of its subsequent quantitative analysis. Moreover, since for an oscillatory non-conservative system all invariant manifolds belong to the stable manifold, defining NNMs only based on the invariance property makes the approach beneficial only for transient analysis.

In this regard, we propose an approach that incorporates both periodicity and invariance properties in an alternative definition of nonlinear modes of vibration for both conservative and non-conservative systems. Then, in Chapter 3, we show that the proposed definition leads to a new method of computing nonlinear modes which is capable of finding solutions that had been remained latent using the other definitions. Through discussion presented herein, we explain that these solutions are the key to unify the mentioned definitions into one framework. In addition, in the following chapters we elucidate why uncovering these solutions are absolutely necessary in order to explain the well known, and exclusively nonlinear phenomena such as bifurcations and jump (or catastrophes in general).

2.1 Definitions of Nonlinear Normal Modes

As mentioned earlier, attempts to extend the concept of linear modes to nonlinear systems led to three different definition of NNMs. Key concepts of these definitions will be reviewed here.
2.1.1 Nonlinear Modes as Periodic Functional Relations

In order to present Rosenberg’s definition, consider a $n$ degrees of freedom (DOF), conservative nonlinear system expressed by

$$\ddot{x} = -\nabla U = f_c(x),$$  \hspace{1cm} (2.1)

where $x \in \mathbb{R}^n$ is the displacement vector, $f_c \in \mathbb{R}^n$ is the vector of nonlinear conservative forces and $U : \mathbb{R}^n \to \mathbb{R}$ is the positive definite potential function. Rosenberg assumed that the potential function is symmetric with respect to the origin in which case the existence of a stable equilibrium at the origin is guaranteed. The imposed symmetry property also excludes nonlinear forces that have quadratic terms, e.g. it excludes the system $\ddot{x} = -x^2$. Rosenberg then defined the normal mode of the system (2.1) as a vector function

$$x = \eta(x_r), \eta_r(x_r) = x_r,$$  \hspace{1cm} (2.2)

where $\eta$

- is satisfied for all $t$ by periodic solutions $x = [x_1, ..., x_r, ..., x_n]^T$ of (2.1).

- satisfies the boundary condition $\eta(0) = 0$, and

- intersects the equi-potential line defined by $U(x) = U_0$, $U_0 \in \mathbb{R}^+$ orthogonally [114].
Figure 2.1: Schematic illustration of the orthogonality in Rosenberg’s definition for a two DOF system with a NNM defined by \( \mathbf{x} = [x_1 \ x_2]^T = \eta(x_1) = [\eta_1(x_1) \ \eta_2(x_1)]^T \). Left: the potential function \( U \) and the equi-potential curve \( U = U_0 \) are shown. Right: The curve \( \eta_2(x_1) \), i.e. the mode shape of a NNM crosses the equi-potential contour, i.e. \( U(x) = U_0 \), of the potential function \( U(x) \) orthogonally.

The first condition describes a mode as a functional relation that defines all coordinates of any periodic solution of the system (2.1) from one, and only one, of its coordinates. The second condition restricts the periodic solutions in the first condition to only synchronous periodic motions. That is because, \( x(0) = \eta(x_r(0)) = 0 \) implies that \( x_i|_{t=0,T} = x_r|_{t=0,T}, i = 1, ..., n, i \neq r \) meaning all the masses pass through the equilibrium state simultaneously. The orthogonality condition is illustrated for a two DOF system in Fig. 2.1.

The relation (2.2) was called the “modal relation” by Rosenberg [117], a terminology that has been retained by others. Each valid solution for the modal relation (2.2) defines a NNM of the system.

2.1.2 Nonlinear Modes as Periodic Solutions

The last two conditions in Rosenberg’s definition are suspected of being added merely for the purpose of providing a way to find periodic solutions of (2.1) analytically, since at the time, no numerical algorithm or computational means for (practical) numerical computation of such solutions were available. In this sense, four decades later, Rosenberg’s definition was modified by Vakakis, Kerschen and others [80, 81] by relaxing the synchronicity, i.e. \( \eta(0) = 0 \), and orthogonal intersection, i.e. \( U(x) = U_0 \), conditions. This allowed the definition to be extended to include cases of internal resonance where coordinates may
not oscillate synchronously. The positive definite and symmetry properties of the potential function were also removed in favor of extending the definition to include all conservative forcing functions [105]. Therefore, the modified Rosenberg definition considers all periodic solutions of the systems as NNMs and vice versa. In this definition, finding the modal relation (2.2), i.e. the mode shapes, is no longer associated with finding the NNMs of the system and NNMs are defined only based on their periodic characteristic such as their (fundamental) frequency. A brief summary of this approach is provided next.

Consider a \( n \)-DOF, nonlinear, conservative vector field \( X_c \) expressed by

\[
\dot{z} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} y \\ f(x) \end{bmatrix} = X_c(z) \tag{2.3}
\]

where \( x, \dot{x} \in \mathbb{R}^n \) and \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is the conservative vector of the nonlinear force. Then following the works of [128, 34, 41], Kerschen et al. recast the problem of finding periodic solutions of (2.3) as searching for solutions of a nonlinear eigenvalue problem defined by

\[
G(z_0, T) = z_0 - \int_0^T X_c(z(t)) dt, \tag{2.4}
\]

in other words, finding the set \( S = \{(z_0, T) | G(z_0, T) = 0, T \in \mathbb{R}^+ \} \) [80, 81].

Assuming that the function \( G(z_0, T) \) satisfies the Implicit Function Theorem (IFT), then it can be shown that a non-singular eigensolution of (2.4), i.e. a solution \((z_0, T)^*\) of \( G(z_0, T) = 0 \) with a non-singular \( J_G(z_0^*, T^*) \), will give rise to a continuous branch of solutions \( z_0(T) \) with \( z_0(T^*) = z_0^* \) (for complete statement and proof see [79, 89, 42]). The existence of a continuous branch of solutions is usually known as persistence of solutions or continuation and is widely used to find the solutions of (2.4) [128, 34, 41]. In this regard, Kerschen et al. used the linear modes of the system (2.3) and their periods as the initial non-singular solution \((z_0, T)^*\) to find up to \( n \) branches of periodic solutions [80, 81].
In this regard, for a two DOF system (shown in Fig. 2.2) defined by

\[
\begin{bmatrix}
\ddot{x}_1 \\
\ddot{x}_2
\end{bmatrix} = - \begin{bmatrix}
2 & -1 \\
-1 & 2
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} - \begin{bmatrix}
0.5x_1^3 \\
0
\end{bmatrix},
\]

one can solve the nonlinear eigenvalue problem (2.4) to find branches of periodic solutions using continuation of linear modes, i.e. \((z_0, T)^* = (z_0, T)|_{\text{Linear Mode}}\), as shown in Fig. 2.3.

Figure 2.3: Two main branches of NNMs obtained by continuation of LNMs. The region revealing cases of internal resonance is magnified.
2.1.3 Nonlinear Modes as Solutions on Invariant Manifolds

In this definition, the concepts of manifold and invariance play crucial roles. Therefore theses concepts are explained, in plain language, next (for more rigorous definitions, readers are refereed to the provided references).

**Manifold** - In applications, the term manifold is used to define $m$-dimensional surfaces embedded in $\mathbb{R}^N$, $m < N$. If the surface has no singular points, i.e. points on the surface that generate non-unique tangent planes\(^5\), then by the implicit function theorem it can locally be represented as a graph\(^6\) \[48\]. In this sense, singular points define the boundaries of a manifold.

**Invariance** - Suppose a manifold $\Gamma$ and a vector field $X$ (defining a $n$-DOF, nonlinear, non-conservative system) expressed by

\[
\dot{z} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} y \\ f(x,y) \end{bmatrix} = X(z) \tag{2.6}
\]

are defined in a space spanned by $x, \dot{x} \in \mathbb{R}^n$ where $f : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is the vector of nonlinear forces. Then, in order to start from a point in this space, i.e., $\bar{z}$, and move on the surface of the manifold $\Gamma$ or along the vector field $X$, one needs to steer along the tangent vector of the invariant manifold or the vector field denoted by $\frac{dT}{dt}$ and $\frac{dz}{dt} = X$, respectively. In this sense, the vector field is said to be operating on the manifold if one uses the tangent direction given by the vector field, i.e., $\frac{dz}{dt} = X$, instead of a vector from the tangent space of the manifold denoted by $\frac{dT}{dt}$, to navigate the points on the invariant manifold. Then, the manifold $\Gamma$ is said to be invariant under the operation of the vector field $X$, if the points on the manifold remain on its surface after moving along the tangent of the vector field. Therefore, one can conclude that a manifold $\Gamma$ is invariant under the operation of a vector field $X$ in the neighborhood of a point $\bar{z}$, if both have the same tangent at $\bar{z}$. This can be stated mathematically as

\(^5\)If the derivative of the function representing the surface does not have maximal rank.
\(^6\)It locally has the structure of an Euclidean space.
\[
\frac{dz}{dt}(\bar{z}) = X(\bar{z}) = \frac{d\Gamma(\bar{z})}{dt}.
\]  
(2.7)

A schematic illustration of this concept is provided in Fig. 2.4.

Figure 2.4: The operation of the vector field \(X\) on the manifold \(\Gamma\). Left: Directing a point \(\bar{z}\) in a space \(\mathbb{R}^{2n}\) under the vector field \(X\). Middle: Moving a point \(\bar{z}\) on the manifold \(\Gamma\). Right: Navigating a point \(\bar{z}\) on the manifold \(\Gamma\) in the direction given by the vector field \(X\).

Using this concept\(^7\), Shaw and Pierre defined a NNM as a motion that takes place on a two-dimensional manifold \(M : \mathbb{R}^2 \to \mathbb{R}^{2n}\) which is tangent to the vector field at a stable equilibrium point \([122, 123]\). It is important to notice that these invariant manifolds are not NNMs but the motions that take place (start) on these manifolds are defined as NNMs. In other words, these invariant manifolds are the loci of NNMs of the system in the state space.

According to (2.7), the tangency property of the manifold \(M\) guarantees the invariance of this manifold \([33, 140, 141, 63]\). In order to parametrize the invariant manifold \(M\), a pair of variables, i.e. the displacement and the velocity, were chosen as independent variables while the remaining coordinates were assumed to functionally depend upon the independent pair. Assuming, without loss of generality, \((x_1, y_1)\) are the independent coordinates, Shaw and Pierre expressed the functional dependence, i.e. the manifold \(\Gamma\), as

\(^7\)The explanation given herein for the concept of invariance of is not mentioned in Shaw and Pierre’s paper.
\[ \mathbf{\Gamma}(x_1, y_1) = [x^T y^T]^T = [x_1, X_2(x_1, y_1), \ldots, X_n(x_1, y_1), y_1 Y_2(x_1, y_1), \ldots, Y_n(x_1, y_1)]^T. \] (2.8)

This surface is tangent to the vector field if the relation

\[ \mathbf{X}(\bar{z}) = \mathbf{X}(\mathbf{\Gamma}(x_1, y_1)) = \frac{d}{dt} \mathbf{\Gamma}(x_1, y_1), \] (2.9)

and from differential geometry \( \frac{d}{dt} \mathbf{\Gamma} \) can be found as

\[ \frac{d}{dt} \mathbf{\Gamma}(x_1, y_1) = \mathbb{J}\mathbf{\Gamma}(x_1, y_1) \begin{bmatrix} \dot{x}_1 \\ \dot{y}_1 \end{bmatrix}, \] (2.10)

where \( \mathbb{J} \) is the Jacobin matrix \([130, 48]\). Therefore, using the differential equations of the vector field \((2.6)\) at the equilibrium\(^8\), one can obtain

\[ \left[ [y_1, Y_2(x_1, y_1), \ldots, Y_n(x_1, y_1)]^T, f^T \right]^T = \mathbb{J}\mathbf{\Gamma}(x_1, y_1) \begin{bmatrix} y_1 \\ f_1 \end{bmatrix}. \] (2.11)

In order to construct the manifold \( \mathbf{\Gamma} \), i.e. to solve the nonlinear algebraic system of equation \((2.11)\), Shaw and Pierre proposed to use a general two-variable Taylor series expansions for all the unknown functional relations \( Y_i, i = 2, \ldots, n \). For instance, using third order Taylor series expansions, Shaw and Pierre calculated two invariant manifolds (associated with the fixed point of) the system \((2.5)\) as

\[ \mathbf{\Gamma}_1(x_1, y_1) = \begin{bmatrix} x_1 + \frac{1}{12}(2x_1^2 + 3y_1^2)x_1 \\ y_1 + \frac{1}{4}y_1^3 \end{bmatrix} + O(x_1^4, y_1^4), \quad \mathbf{\Gamma}_2(x_1, y_1) = \begin{bmatrix} x_1 \\ -x_1 + \frac{1}{120}(10x_1^2 + 3y_1^2)x_1 \end{bmatrix} + O(x_1^4, y_1^4). \] (2.12)

### 2.1.4 A Note on Invariance and Periodicity of Nonlinear Modes

Merely based on the domains of the given definitions of nonlinear modes, one can portray the relationship between them that as in Fig. 2.5. However, in this section, we wish to further explore the relationship

\(^8\)It is assumed that the equilibrium of the system is translated to the origin.
between the set of periodic motions and the set of invariant solutions. In other words, we seek to answer two questions; does any periodic solution of an oscillatory system belongs to an invariant manifold? Conversely, are any of the solutions initiated on an invariant manifold is periodic? In this regard and since non-conservative systems do not accept periodic solutions, we focus our attention on conservative systems.

![Figure 2.5: Domain of definitions of nonlinear modes for oscillatory systems.](image)

To answer the first question, we shall restate it more precisely. Any periodic solution generates a closed curve in the state space on which it repeats its path and never separates from it. In this sense, any periodic solution can be viewed as a one-dimensional invariant manifold [71, 91]. Therefore, the first question can (should) be restated as: is there a set of periodic solutions (one-dimensional invariant manifolds) that constructs a two-dimensional invariant manifold (i.e. a Shaw and Pierre invariant manifold)? This, according to the definition of the invariant manifold, means that the geometry constructed by the solutions in (the locus of) such a set must have no singular point. This point is the key to connect the two formulations stated in equations (2.11) and (2.4). As shown in [79, 89, 125, 42], singular points on any branch of periodic solutions, i.e. any point \((z_0^*, T^*)\) with \(J_G(z_0^*, T^*) = 0\), are the only singular points on the geometry constructed by the periodic solutions, i.e. any point along the solution \(z^*\) with \(J_{\Gamma(z^*)} = 0\). In other words, if one breaks down a branch of periodic solutions into a set of sub-branches at its singular points, each sub-branch constructs a two-dimensional invariant
manifold in the state space. This fact can also be used to construct two-dimensional invariant manifolds from periodic solutions of a conservative system. In this regard, as illustrated in Fig. (2.6), one can use periodic solutions as building blocks (stripes) to create a mesh that represents the geometry of the manifold. In this figure, the set of periodic solutions are stacked on top of each other, where each stripe corresponds to a set of consecutive periodic solutions on a subsection of a continuous branch of periodic solutions.

Figure 2.6: A singular-point-free section of the second branch of periodic solutions from Fig. 2.3 is shown in the lower right plot. The two-dimensional invariant manifold representation of this branch is shown in the upper left plot. This branch is divided into seven subsections colored in red and black and connected to their corresponding regions on the invariant manifold in the lower left plot. Regions associated to the red subsections are shown in the upper middle and right plots. One can observe that linear modes (with low energy $< 1$) correspond to an almost flat plane.

We answer the second question by explaining that it’s contra-positive equivalent is true. If the independent set of variables is periodic so is the dependent set, i.e. the solutions on the invariant manifold. However, if the solution of system for the independent set, in a representation of an invariant manifold,
is not periodic, the resultant functional representation is not periodic either. For conservative systems, the solution of the system (including the independent set) is in general an almost periodic function\(^9\) (almost periodicity and key concepts of relevance are briefly discussed in Appendix 1). Therefore, since any function of almost periodic variable(s) is almost periodic [19, 112, 97], the solutions on the invariant manifolds are in general almost periodic. Therefore, in the same way that periodic solutions are a special subset of all almost periodic solutions for the independent set, periodic solutions on the invariant manifolds of the system are only a subset of all solutions that may be initiated on the invariant manifolds.

Figure 2.7: (a) Two synchronous periodic motions (in blue and cyan) and one non-synchronous periodic motion (in red) of the system in (2.5). (b)-(d) An almost periodic motion (in black) is added and simulated for ascending duration time. Almost periodic motions, unlike periodic ones, do not form close curves. Instead, they form dense sets, wherein, their curve can get arbitrary close to every point in the set as demonstrated in plots (b)-(d).

2.1.5 A Note on Shaw and Pierre’s Invariant Manifolds

Shaw and Pierre’s invariant manifold approach to define nonlinear modes has three main limitations. First, it defines the invariant manifolds only for a special case of equilibrium of the system, i.e. fixed

---
\(^9\)Almost periodic functions are bounded functions that come arbitrary close to retrace their path in the state space but never form closed curves. These solutions construct dense sets in the state space wherein they can get arbitrary close to any point in the set. One can easily construct an almost periodic function by adding two periodic functions with incommensurable periods. Two numbers \(T_1\) and \(T_2\) are said to be incommensurable if their ratio \(\frac{T_1}{T_2}\) is an irrational number.
points and ignores manifolds that stem from periodic orbits (examples of these manifolds shall be shown subsequently). This exclusion restricts the definition’s ability to capture all the (potentially infinitely many) invariant manifolds. For instance, as mentioned earlier, any \( n \)-DOF Hamiltonian system has at least at least \( n \) periodic solutions (normal modes) passing through each stable equilibrium which, subjected to satisfy some conditions, start branches of periodic orbits. To further illustrate this, Fig. 2.8 shows that while synchronous periodic motions lie on the Shaw and Pierre invariant manifold, the non-synchronous periodic motion passes through the invariant manifold.

Figure 2.8: Left: Two synchronous periodic motions, i.e. Rosenberg NNMs, and one non-synchronous periodic motion, a modified Rosenberg’s NNM, both starting from the invariant manifold of the system. Top-Right: Both synchronous periodic motions lie on the surface of the invariant manifold, i.e. the locus of Shaw and Pierre’s NNMs, while the non-synchronous periodic motion passes through the invariant manifold. Bottom-Left: The \( x_2 - x_1 \) plane view of the invariant manifold reveals the mode shape of the non-synchronous periodic motion.

It is important to notice that this phenomena can not be traced back to computational error or lack of higher order terms in the polynomial approximation. Because, by definition, the invariant manifold is tangent to the vector field (and linear normal modes planes) at the equilibrium, it has to be planar close to the equilibrium. This can be easily observed in Fig. 2.6. In contrast, the non-synchronous periodic motion flows on the surface of a torus. This is shown in the bottom-right plot of Fig. 2.8. The invariant manifolds of an equilibrium are defined only locally around the equilibrium of the system.
For invariant manifolds of a fixed point, the approach proposed by Shaw and Pierre may be applied easily since the origin is usually the fixed point of the system or it can be transferred to the origin through a simple change of variables. However, for periodic orbits (as equilibria), one needs to find the representation of the system around the period orbit before applying the approach\textsuperscript{10}. However, this is not possible without first finding the periodic solution. Moreover, as shown in Fig. 2.9, using a general Taylor series expansion even for fixed points does not lead to an accurate expression unless very high orders are included (see Fig. 2.9) which would exponentially increase the number of coefficients and equally the size of the nonlinear system of algebraic equations that must be solved to find those coefficients.

Figure 2.9: Comparison between invariant manifolds constructed using periodic motions and Pierre and Shaw’s Taylor series approach. Top: Periodic motions in the first sub-section of the plot in Fig. 2.6 are used for both constructing a manifold (as a geometric mesh) and the independent set \((x_1, \dot{x}_1)\) in Pierre and Shaw invariant manifold presented in Eq. (2.12). A third order approximation of the Pierre and Shaw’s manifold (top-middle), even for the very low energy solutions (almost linear), do not coincide with the exact manifold (top-right: constructed as geometric mesh using periodic orbits). Bottom: The deviation of Pierre and Shaw’s manifold (bottom-middle) from the true manifold (bottom-right) becomes more evident as one adds the second sub-section from the plot in Fig. 2.6.

\textsuperscript{10}Finding the Jet-\(k\) reduction of the system around the periodic solution.
Furthermore, the region of validity in Shaw and Pierre’s approach, i.e. the boundary of the manifolds, is not defined. In other words, using this approach, one can not identify a region such as the inside of the closed curve defined by $R(x_1, \dot{x}_1) = 0$, wherein the surface $\Gamma$ defines a manifold and is invariant. We will explain this further in Section 4.

2.2 Instantaneous Center Manifold (ICM)

2.2.1 Background; Solutions of Nonlinear Vector Fields Close to Equilibrium

The concept of invariance in mathematics is almost two centuries old [135]. For a dynamical system defined by the vector field

$$\dot{x} = X(x), \quad x \in \mathbb{R}^{2n},$$

(2.13)

with a fixed point $\bar{x}(t) \in \mathbb{R}^N$ the linear system

$$\dot{y} = Jy, \quad y \in \mathbb{R}^{2n}$$

(2.14)

where $J_X(\bar{x})$ is the Jacobean of the vector field $X$ evaluated at $\bar{x}$, the eigenvectors of $J_X(\bar{x})$ provide a basis for $\mathbb{R}^{2n}$ [72] so that $\mathbb{R}^{2n}$ can be represented as the direct sum of three subspaces denoted by $E^s$, $E^u$, and $E^c$:

$$E^s = \text{span}\{e_1, ..., e_s\}$$

$$E^u = \text{span}\{e_{s+1}, ..., e_{s+u}\}, \quad s + u + c = 2n,$$

$$E^c = \text{span}\{e_{s+u+1}, ..., e_{2n}\}$$

where $\{e_1, ..., e_s\}$, $\{e_{s+1}, ..., e_{s+u}\}$ and $\{e_{s+u+1}, ..., e_{2n}\}$ are the generalized eigenvectors corresponding to the eigenvalues of $J_X(\bar{x})$ having respectively negative, positive and zero real parts. $E^s$, $E^u$ and $E^c$ are examples of invariant subspaces and referred to as the stable, unstable and center
subspaces, respectively. They are invariant because solutions of (2.14) with the initial condition entirely contained in either of them remain in that particular subspace for all time [141, 72, 21]. Since, solutions starting in $E^s$ decay to zero asymptotically as $t \to +\infty$ and solutions starting in $E^u$ approach zero asymptotically as $t \to -\infty$, the invariance property, in fact, helps us to understand the nature of infinitely many solutions of the system through studying a finite set of solutions, i.e. the eigensolutions of (2.14).

Similarly, the concept of invariance can provide the same information about the nature of solutions near the fixed points $x = \bar{x}$ of the original nonlinear system (2.13). In order to study the properties of the solutions of a nonlinear system, the first step is to locally replace it by an equivalent linear system. This equivalence can be precisely expressed by a continuous bijection with a continuous inverse function between the solutions of the two systems. Such a function is called a homeomorphism. If the mentioned function is differentiable, then it’s called a diffeomorphism. The required level of fidelity of the linear system to the original nonlinear system depends on the type of the equilibrium of the nonlinear system around which the system is being studied. In this regard, Grobman [61] and Hratman [68] separately showed that for hyperbolic equilibrium $\bar{x}$ the linearization system (2.14) can effectively can be used to study the solution of the nonlinear system (2.13). The results of their study were later conformed in a theorem sometimes called the Stable and Unstable Invariant Manifold theorem or Hartman-Grobman theorem [63]. A more general case, including a non-hyperbolic equilibrium $\bar{\bar{x}}$, was first studied by Sternberg in a set of three papers [131, 132, 133]. These works share the same results with Hratman-Grobman theorem for hyperbolic equilibrium. However, for non-hyperbolic equilibrium, the equivalence exists only if the nonlinear system can be truly transformed to a linear one, namely the normal form, by a change of coordinates, i.e. a nonlinear transformation. Such equivalence exists in form of a diffeomorphism if and only if
\[
\lambda_i \neq \sum m_j \lambda_j, \quad \forall m_j \in \mathbb{Z}^+, \ m_j > 1, \ i, j = 1, \ldots, 2n
\]  
(2.15)

where \(\lambda_1, \ldots, \lambda_{2n}\) are eigenvalues of \(J_X(\bar{x})\) \([132]\).

However, the existence of the equivalence relation is usually stated under one theorem, namely, the
stable, unstable and center manifold theory and the differentiation between the equivalence relations
for different types of equilibrium is made through follow-up remarks or notes. For instance, in \([63]\),
it is stated in the Hartman-Grobman and the Stable-Unstable Manifold theorem that the behavior of
the solutions near a hyperbolic equilibrium can be studied by the dynamics of the linearization of the
nonlinear system. However, for the more general case, i.e. non-hyperbolic equilibrium, the reader is
referred to Sternberg’s theorem and the chapter that discusses the normal form theory. In another
case, Wiggins \([141]\) points out the mentioned difference after stating the stable, unstable and center
manifold theory in the followup remarks.

In this regard, one can translate the fixed point to the origin \((y = x - \bar{x})\) and find a linear
transformation \(T\) to transform the linear part of Taylor’s expansion of \(X(\bar{x} + y)\) about \(x = \bar{x}\) into
the block diagonal form \([72, 77]\)

\[
\begin{pmatrix}
\dot{u} \\
\dot{v} \\
\dot{w}
\end{pmatrix} =
\begin{pmatrix}
J_s & 0 & 0 \\
0 & J_u & 0 \\
0 & 0 & J_c
\end{pmatrix}
\begin{pmatrix}
u \\
v \\
w
\end{pmatrix}.
\]

Here \(T^{-1} y \equiv (u^T, v^T, w^T)^T \in \mathbb{R}^s \times \mathbb{R}^u \times \mathbb{R}^c\), and \(J_s, J_u\) and \(J_c\) are square matrices having eigenvalues
with negative, positive and zero real parts, respectively. Using the same transformation to transform
the coordinates of the nonlinear vector field (2.13) yields

\[
\begin{align*}
\dot{u} &= J_s u + R_s(u, v, w) \\
\dot{v} &= J_u v + R_u(u, v, w) \\
\dot{w} &= J_c w + R_c(u, v, w),
\end{align*}
\]  
(2.16)
where \( R_s, R_u \) and \( R_c \) are higher order terms of Taylor’s expansion of \( X(\bar{x} + y) \) about \( x = \bar{x} \) transformed by the same linear transformation \( T \) \cite{21,72}.

**Theorem 2.1-** (Local Stable, Unstable and Center Manifolds of Fixed Points) Suppose \((2.16)\) is a \( C^r \) \((r \geq 2)\) vector field. Then the fixed point \((u, v, w) = (0, 0, 0)\) of \((2.16)\) possesses a \( C^r \) \(s\)-dimensional local, invariant stable manifold, \( W_{loc}^s(0) \); a \( C^r \) \(u\)-dimensional local, invariant unstable manifold, \( W_{loc}^u(0) \); and a \( C^r \) \(c\)-dimensional local, invariant center manifold, \( W_{loc}^c(0) \), all intersecting at \((u, v, w) = (0, 0, 0)\). Theses manifolds are all tangent to the respective invariant subspaces of the linear vector field \((2.14)\) at the origin and, hence, are locally representable as graphs. In particular,

\[
W_{loc}^s(0) = \{ (u, v, w) | v = h_v^s(u), w = h_w^s(u); J_{h_v^s}(0) = 0, J_{h_w^s}(0) = 0; \|u\|_2 \text{ sufficiently small} \}
\]

\[
W_{loc}^u(0) = \{ (u, v, w) | u = h_u^u(v), w = h_w^u(v); J_{h_u^u}(0) = 0, J_{h_w^u}(0) = 0; \|v\|_2 \text{ sufficiently small} \}
\]

\[
W_{loc}^c(0) = \{ (u, v, w) | u = h_u^c(w), v = h_v^c(w); J_{h_u^c}(0) = 0, J_{h_v^c}(0) = 0; \|w\|_2 \text{ sufficiently small} \},
\]

(2.17)

where \( h_v^s(u), h_w^s(u), h_u^u(v), h_w^u(v), h_u^c(w), \) and \( h_w^c(w) \) are \( C^r \) functions. Moreover, trajectories in \( W_{loc}^s(0) \), \( W_{loc}^u(0) \) and \( W_{loc}^c(0) \) (subject to some additional conditions \cite{141,63}) have the same asymptotic properties as trajectories in their respective invariant subspaces.

Proof. See \cite{140,72,71}.

The first part of Theorem 2.1, which addresses the existence of the invariant manifolds, is proven for all types of equilibrium. However, the second part, i.e., the equivalence of the asymptotic behavior of the solutions, requires some clarifications. As it is explained in \cite{141,63}, if the equilibrium is hyperbolic\footnote{An equilibrium without any purely imaginary eigenvalues, or equivalently an equilibrium that does not have a center manifold is called a hyperbolic equilibrium.}, the asymptotic behavior of the solutions of the nonlinear system can be described by the behavior of their counterparts on the stable or unstable manifolds since there exist an equivalence relation between the nonlinear system and its linearization. However, for non-hyperbolic equilibrium, such an equivalence relation can not be defined through linearization of the nonlinear system but by a transformation of...
the system of coordinates into a system in which the nonlinear system has a linear representation. Finding such a linear representation is the subject of the normal form theory. Hence, the conditions that determine whether one can use the solutions on the center manifold of a nonlinear system to describe the asymptotic behavior of its solutions, have to come from the normal form theory. Sternberg described these conditions in [132].

2.2.2 Toward ICM; Almost Periodicity of Solutions on Invariant Manifolds

In this section we begin by assuming that some “basic” invariant manifolds exist, then we develop a series of theorems that will explain the properties of the solutions on these manifolds. These theorems will help us to extract a set differential equations governing all the solutions on these manifolds and later to find the geometry of such manifolds. The basic invariant manifolds typically considered are invariant manifolds for (a) equilibrium (fixed) points, and (b) periodic orbits. Analytical examples for both types are given in [77, 134, 141].

Riesz-Fisher Theorem- Any trigonometric series defined by \( \sum_{n=0}^{\infty} A_n e^{i \Lambda_n t} \), where \( \Lambda_n \in \mathbb{R} \) and \( A_n \in \mathbb{C} \), subject to the condition that the series \( \sum_{n=0}^{\infty} \| A_n \|^2 \) is convergent, is the generalized Fourier series of an almost periodic function \( f(t) \).

Proof. See [19, 20].

In the next two theorems, we prove the almost periodicity of the stable solutions of a nonlinear system close to its equilibrium by connecting it to almost periodicity of the solutions on the center subspace of its linear representation. This is done under conditions stated in [132] and by using Theorem 2.1 and the Riesz-Fisher theorem.

Theorem 2.2- (Almost Periodicity of Solutions on Center Manifolds) Suppose \( \mathbf{X} \) is a \( C^r \) vector field in \( \mathbb{R}^N \) vanishing at \( \bar{x} \). If \( \bar{x} \in W^s_{\bar{x}} \), where \( W^s_{\bar{x}} \) is defined in (2.17), and if for all eigenvalues \( \lambda_1, ..., \lambda_N \) of

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12In this paper, the aforementioned conditions are given for two types of representations of dynamical systems, namely maps (in Theorem 1) and differential equations (in Theorem 2).
\[ J_X(\bar{x}), \text{ the condition} \]

\[ \lambda_i \neq \sum m_j \lambda_j, \forall m_j \in \mathbb{Z}^+, m_j > 1 \quad (2.18) \]

holds, then all solutions close to \( \bar{x} \), i.e. \( x \) are locally almost periodic.

Proof. Sternberg [132] proved that under the assumptions stated, there exists a change of coordinates \( T \) that provides a system of coordinates wherein \( X \) has a linear representation. Therefore, by Theorem 2.1, the trajectories of the vector field \( X \) can be reproduced by trajectories of the invariant subspaces of its linear representation. Hence, it follows from condition (2.18) and Theorem 2.1 that, locally, \( x \in \mathbb{E}^c \). Since [71] have proved that \( \mathbb{E}^c = \text{span} \{ e_{s+u+1}, \ldots, e_{s+u+c} \} \), the relation

\[ x = e^{[P_c \Lambda_c P_c^T]}_t x_0 = \sum_{i=s+u+1}^{s+u+c} e_i e_i^T x_0 e_i t, \]

holds for

\[ P_c = [e_{s+u+1} \ldots e_{s+u+c}], \quad \Lambda_c = \text{diag} (\lambda_{s+u+1}, \ldots, \lambda_{s+u+c}). \]

Therefore,

\[ \sum_{i=s+u+1}^{s+u+c} \| e_i e_i^T x_0 \|^2 = \sum_{i=s+u+1}^{s+u+c} (e_i e_i^T x_0)^T (e_i e_i^T x_0) = \sum_{i=s+u+1}^{s+u+c} x_0^T [e_i (e_i^T e_i)] x_0 = \sum_{i=s+u+1}^{s+u+c} x_0^T [e_i e_i^T] x_0 = x_0^T P_c P_c^T x_0 = \| x_0 \|^2. \]

Then, from the Riesz-Fisher theorem, it follows that \( x \) is almost periodic.\[ \blacksquare \]

Condition (2.18) defines the geometric boundaries of the center manifold. That is because if it does not hold, the Jacobian of the vector field will be singular. This Jacobian, as it will be discussed in Section 3, defines the tangent on the surface representing the center manifold and also is crucial in determining the invariance of it. Therefore, if the Jacobian is singular then, by the definition 2.2, the candidate surface can not be locally represented by a graph. Consequently, it can not represent a
manifold. Moreover, since the candidate surface can not represent a manifold, its invariance can not be defined.

**Theorem 2.3** (Asymptotic Almost Periodicity of Stable Solutions) Suppose $X$ is a $C^r$ vector field in $\mathbb{R}^N$ vanishing at $\bar{x}$. If $\bar{x}$ is not unstable and the condition equation (2.18) holds, then locally, all solutions close to $\bar{x}$, i.e. $x$, are asymptotically almost periodic.

**Proof.** Since $\bar{x}$ is not unstable and condition (2.18) holds, it follows from Sternberg’s theorem and Theorem 2.1 that

$$x \in E^{s+u+c} = E^s \oplus E^c = \text{span} \{ e_1, \ldots, e_s, e_{s+u+1}, \ldots, e_{s+u+c} \}.$$ 

Therefore,

$$x = e^{[P_{s\oplus c} \Lambda_{s\oplus c} P_{s\oplus c}^T]t} x_0,$$  \hspace{1cm} (2.19)

where

$$P_{s\oplus c} = [e_1 \ldots e_s \ e_{s+u+1} \ldots e_{s+u+c}], \quad \Lambda_{s\oplus c} = \text{diag} (\lambda_1, \ldots, \lambda_s, \lambda_{s+u+1}, \ldots, \lambda_{s+u+c}).$$

One can simplify (2.19) to

$$x = P_{s\oplus c} \Lambda_{s\oplus c}^t P_{s\oplus c}^T x_0,$$

in which $\Lambda_{s\oplus c}^t = e^{(A_{s\oplus c})t}$. Then

$$x = [P_s \Lambda_s^t P_s^T + P_c \Lambda_c^t P_c^T] x_0.$$

Since $\lim_{t \to \infty} \Lambda_s^t \to 0$, for large values of $t$, $\tilde{x}(t)$ converges uniformly to $P_c \Lambda_c^t P_c^T x_0$ which by Theorem 2.2, is an almost periodic function. ■
The next theorem, assuming that one could reduce the center manifold to a set of low dimensional invariant manifolds, states the general expression of solutions on these manifolds and consequently the general expression of the solutions on the center manifold and of the nonlinear system.

**Theorem 2.4** Suppose $X$ is a $C^r$ vector field in $\mathbb{R}^{2N}$ for the system (2.6) vanishing at $\bar{x}$. If the solution $\bar{x}$ is not unstable, and the condition defined by (2.18) holds for the linear system associated with the vector field $X$ along the solution $\bar{x}$, then for any subset of coordinates such as $\{ (x_i, y_i) | i = 1, \ldots, k < N \}$, all other coordinates $\{ (\bar{x}_j, \bar{y}_j) | j \neq i \}$ of the solution $x$ of the linearized system, can be asymptotically expressed in the form

$$
\begin{align*}
  x_j &= \sum_{m \in \mathbb{Z}^k} C^{(i)}_m e^{i \langle m, u \rangle} \\
  y_j &= \nabla_u x_j \cdot v = D^u_x x_j \\
  &\quad \quad m \in \mathbb{Z}^k, \ C^{(i)}_m \in \mathbb{R}, \ |m| \geq N - k \\
  u &= (\phi_1, \ldots, \phi_k), \ v = (\dot{\phi}_1, \ldots, \dot{\phi}_k)
\end{align*}
$$

where $\dot{\phi}_j$ is given by (2.27), $m = (m_1, \ldots, m_k)^T$ and $\langle m, u \rangle = \sum_{i=1}^k m_i u_i = \sum_{i=1}^k m_i \phi_i$.

**Proof.** Suppose $X_l(\bar{x})$ is a linear system associated with the vector field $X$. Since $\bar{x}$ is not unstable and the condition (2.18) holds any solution of $X_l(\bar{x}^*)$, i.e. $x$, is an asymptotically almost periodic function and therefore it asymptotically belongs to the space of almost periodic functions $\mathcal{B}$, which is by definition a pre-Hilbert space [100, 20]. Let $(e_m)_{m \in \mathbb{Z}^k}$ be a family in a pre-Hilbert space defined by

$$
e_m : t \mapsto \frac{1}{(2\pi)^{\frac{k}{2}}} e^{i \langle m, u \rangle}.
$$

Therefore defining the inner product

$$
\langle e_m, e_n \rangle = \frac{1}{(2\pi)^k} \int_{-\pi}^{\pi} e^{i \langle (m+n), u \rangle} du_1 \cdots du_k
$$

it follows that
\[
\langle e_m, e_n \rangle = \frac{1}{(2\pi)^k} \int_{-\pi}^{\pi} e^{i(x+y)u} du = \frac{1}{(2\pi)^k} \left( \int_{-\pi}^{\pi} e^{im_1u} e^{in_1u} du \right) \times \cdots \times \left( \int_{-\pi}^{\pi} e^{im_ku} e^{in_ku} du \right)
\]

\[
= \frac{1}{(2\pi)^k} \left( \int_{-\pi}^{\pi} [\cos(m_1u_1) \cos(n_1u_1) + 2i \cos(m_1u_1) \sin(n_1u_1)] du \right) \times \cdots \times \left( \int_{-\pi}^{\pi} [\cos(m_ku_k) \cos(n_ku_k) + 2i \cos(m_ku_k) \sin(n_ku_k)] du \right)
\]

\[
= \frac{1}{(2\pi)^k} \left[ \pi \delta_{m_1n_1} + 0 + \pi \delta_{m_1n_1} \right] \times \cdots \times \left[ \pi \delta_{m_kn_k} + 0 + \pi \delta_{m_kn_k} \right]
\]

\[
= \frac{1}{(2\pi)^k} \left( 2\pi \delta_{m_1n_1} \right) \times \cdots \times \left( 2\pi \delta_{m_kn_k} \right) = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases} = \delta_{mn}
\]

Therefore \((e_m)_{m \in \mathbb{Z}^k}\) is an orthonormal system in \(\mathbb{B}\). Thus, for every finite subset \(M \subset \mathbb{N}\), \(P_M : x \rightarrow \sum_{i \in M} < x, e_{m_i} > e_{m_i}\) is an orthogonal projection of \(\mathbb{B}\) onto \(\mathbb{B}_M := \text{span}\{(e_{m_i})_{m_i \in \mathbb{Z}^k} : i \in M\}\).

Hence,

\[
\|x\|^2 = \|P_Mx\|^2 + \|x - P_Mx\|^2.
\]

Moreover,

\[
\|\sum_{i \in M} \lambda_i e_{m_i}\|^2 = \langle \sum_{i \in M} \lambda_i e_{m_i}, \sum_{j \in M} \lambda_j e_{m_j} \rangle = \sum_{i,j \in M} \lambda_i \lambda_j^* \langle e_{m_i}, e_{m_j} \rangle = \sum_{i \in M} |\lambda_i|^2,
\]

which proves the linear independence of the family \((2.21)\), it follows that

\[
\|x\|^2 = \sum_{i \in M} |< x, e_{m_i} >|^2 + \|x - P_Mx\|^2.
\] (2.23)

Next, to obtain \((2.20)\), one needs to show that the second part of this representation of \(\|x\|^2\) is equal to zero. This is necessary because the orthonormal system provided in \((2.21)\) is an incomplete basis for \(\mathbb{B}\) [20, 112]. For complete orthonormal systems, such as \(\{e_i\}_{i=1}^{k}\) in \(\mathbb{R}^k\), proving \(\|x - P_Mx\|^2 = 0\) is not necessary since it reduces Pythagoras’s equality \((2.23)\) to Parseval’s equation which is proven for all orthogonal projections on any complete basis.

**Lemma 2.4.1** - For every non-zero orthogonal projection \(P\) in a pre-Hilbert space \(\mathbb{B}\)

\[
\|x - Px\| = \text{dist}(x, R(P)) \quad \forall x \in \mathbb{B},
\]
where $R(P)$ is the range of $P$.

Proof. See [97, 84].

Therefore,

$$0 \leq \|x\|^2 - \sum_{i \in M} < x, e_m > |^2 = \|x - P_M x\|^2 = \text{dist}(x, P_M x)^2.$$

Since by Weierstrass’s approximation theorem $\text{span}\{e_m : m \in \mathbb{Z}\}$ is dense in $\mathcal{B}$ ([97]), there exists a finite subset $M_0 \subset \mathbb{Z}$ and a $y \in \mathcal{B}_{M_0}$ with $\|x - y\| \leq \epsilon$. If $M \supset M_0$ is an arbitrary finite subset of $\mathbb{Z}$, then

$$0 \leq \|x\|^2 - \sum_{i \in M} < x, e_m > |^2 = \|x - P_M x\|^2 = \text{dist}(x, P_M x)^2 = \text{dist}((x, \mathbb{B}_M)^2 \leq \|x - y\|^2 \leq \epsilon^2.$$

This proves that there exists a projection $P_M$ for which $\|x - P_M x\|^2$ is arbitrarily close to zero. From this, it follows that

$$\|x\|^2 = \sum_{i \in M} < x, e_m >|^2,$$

holds for each $x \in \mathcal{B}$. Hence (2.23) reduces to $x = P_M x$, for each $\bar{x}_j \in \mathcal{B}$ or

$$\bar{x}_j = \sum_{m \in \mathbb{Z}} C_m^{(j)} e_m = \sum_{m \in \mathbb{Z}} C_m^{(j)} e^{-i <m \cdot u >},$$

where

$$C_m^{(j)} = < \bar{x}_j, e_m > = \frac{1}{\pi^k} \int_{H_k} \bar{x}_j e^{-i <m \cdot u >} du_1 \cdots du_k,$$

and consequently

$$\ddot{y}_j = \frac{d}{dt} \dot{x}_j = \sum_{i=1}^k \frac{\partial}{\partial u_i} \dot{x}_j \frac{d}{dt} u_i = \nabla_u \dot{x}_j \cdot v \equiv D_u \dot{x}_j.$$


2.2.3 Definition of ICM

In the previous section, the general form for the solutions on a set of low dimensional invariant manifolds was provided. This section, using an approach based on the center manifold theory, characterizes the dynamics of the solutions on these manifolds by a set of differential equations extracted from the EOM of the original nonlinear system. Then, this set of differential equations can be solved by using the provided general form. The obtained solutions can be further used to construct the geometry of these manifolds.

In this regard, a new augmented system is defined based on the system

\[
\begin{bmatrix}
\dot{x} \\
\dot{y}
\end{bmatrix} = \begin{bmatrix}
y \\
f(x, y)
\end{bmatrix}
\] (2.25)

by

\[
\begin{aligned}
\dot{x} &= y \\
\dot{w} &= z \\
\dot{y} &= f(x, y) \\
\dot{z} &= H_f(x, y)
\end{aligned}
\] (2.26)

where \( w = H_x, z = H_y \) in which \( H_v \) represents the Hilbert transform of the variable \( v \). This system, using the transformation

\[
\begin{aligned}
a_i(t) &= (x_i^2 + w_i^2)^{\frac{1}{2}} \\
x_i(t) &= a_i \cos \phi_i \\
y_i(t) &= a_i \cos \phi_i - a_i \omega_i \sin \phi_i \\
\phi_i(t) &= \arg(w_i, x_i)
\end{aligned}
\] (2.27)

where \( i = 1, ..., n \), can be transformed into instantaneous coordinates as

\[\text{\footnote{It is worth mentioning that, although this augmentation doubles the size of the system, it does not change the underlying linear eigenspace of the system.}}\]
\[ \dot{\phi} = \omega \]
\[ \ddot{a} = \rho \]
\[ \dot{\omega} = \text{diag}\left( \frac{\cos \phi_i}{a_i} \right)Hf - \text{diag}\left( \frac{\sin \phi_i}{a_i} \right)f - 2\text{diag}\left( \frac{\rho_i \omega_i}{a_i} \right)\bar{e} \]
\[ \dot{\rho} = \text{diag}(\sin \phi_i)Hf + \text{diag}(\cos \phi_i)f + \text{diag}(a_i \omega_i)\bar{e}, \]

where \( \bar{e} = [1, ..., 1]^T \in \mathbb{R}^n \) (see Appendix 2).

To this end, it is assumed that there exists at least one solution \( \bar{x} \) for which all coordinates are functionally related to a set of paired coordinates which, without loss of generality, is assumed to be \( (x_1, ..., x_k, y_1, ..., y_k), k < N \). This way, the system in (2.25) can be expressed based on coordinates of the system in (2.28), in other words

\[ x_i = a_i \cos \phi_i \]
\[ y_i = \dot{a}_i \cos \phi_i - a_i \omega_i \sin \phi_i \]

and

\[ x_i = U_i(u, r), \quad i = k + 1, ..., N, \]
\[ y_i = V_i(u, v, r, s) \]

where

\[ u = (u_1, ..., u_k) = (\phi_1, ..., \phi_k), \]

and similarly \( v = \omega, \ r = a \) and \( s = \dot{a}. \) Likewise, the vector \( f \) can be reformulated as

\[ f_i = g_i(u, v, r, s), \quad i = 1, ..., N. \]

Equation (2.30) defines a manifold (constraint surface) of dimension \( 2k \) or co-dimension \( 2(N - k) \). A manifold is invariant under a vector field close to its equilibrium, if it is tangent to the vector field at
that equilibrium. This is often called the tangency condition and it guarantees the invariance of the manifold \([33, 140, 141, 63]\). In order to impose the tangency condition to a manifold defined by

\[\Gamma = \begin{bmatrix} U_i(u, r) \\ V_i(u, v, r, s) \end{bmatrix} \] (2.32)

where its Jacobian is defined by

\[\mathbb{J}_{\Gamma}(u,v,r,s) = \begin{bmatrix} \frac{\partial U_i}{\partial u_j} & \frac{\partial U_i}{\partial r_j} & 0 & 0 \\ \frac{\partial V_i}{\partial u_j} & \frac{\partial V_i}{\partial v_j} & \frac{\partial V_i}{\partial v_j} & \frac{\partial V_i}{\partial s_j} \end{bmatrix}, \] (2.33)

and its tangent given by

\[\frac{d}{dt}\Gamma = \mathbb{J}_{\Gamma}(u,v,r,s) \begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{r} \\ \dot{s} \end{bmatrix}, \] (2.34)

the manifold \(\Gamma\) has to satisfy

\[\frac{d}{dt} \begin{bmatrix} U_i(u, r) \\ V_i(u, v, r, s) \end{bmatrix} = \frac{d}{dt} \Gamma = \mathbb{J}_{\Gamma}(u,v,r,s) \begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{r} \\ \dot{s} \end{bmatrix}. \] (2.35)

One can also directly conclude from (2.28) that

\[\dot{v} = \text{diag}(\cos u_i)H_g - \text{diag}(\sin u_i)g - 2\text{diag}(s_i v_i)\bar{e} \equiv F_v \]

\[\dot{s} = \text{diag}(\sin u_i)H_g + \text{diag}(\cos u_i)g + \text{diag}(r_i v_i^2)\bar{e} \equiv F_s \] (2.36)

where \(g = [g_1, ..., g_k]^T\) and \(\bar{e} = [1, ..., 1]^T \in \mathbb{R}^k\).

This way, the set of equations in (2.35), can be solved for \(U_i\) and \(V_i\) to obtain \(2N - 2\) equations\(^{14}\)

\(^{14}\)In a different approach, one could introduce time dependence to \(U_i\) and \(V_i\) by differentiating with respect to time and then substituting into the EOM to obtain a set of differential equations similar to (2.37).
\[ V_i(u, v, r, s) = \sum_{j=1}^{k} \left[ \frac{\partial U_i}{\partial u_j} v_j + \frac{\partial U_i}{\partial r_j} s_j \right] \]
\[ g_i(u, v, r, s) = \sum_{j=1}^{k} \left[ \frac{\partial V_i}{\partial u_j} v_j + \frac{\partial V_i}{\partial v_j} \dot{v}_j + \frac{\partial V_i}{\partial r_j} s_j + \frac{\partial V_i}{\partial s_j} \dot{s}_j \right] \]

\[ i = k + 1, \ldots, N. \]

Given the expression of \( U_i \) in (2.30),

\[ \frac{\partial U_i}{\partial v_j} = \frac{\partial U_i}{\partial s_j} = 0, \quad i = 1, \ldots, N \text{ and } j = 1, \ldots, k. \]

Defining the directional gradient of \( U_i \) by

\[ D^v_u U_i = \nabla_u U_i \cdot v, \]

where \( \nabla_u U_i \) is the partial gradient of the scalar function \( U_i \) with respect to the vector \( u \), the first equation of (2.37) can be reduced to

\[ V_i(u, v) = \sum_{j=1}^{k} \left[ \frac{\partial U_i}{\partial u_j} v_j + \frac{\partial U_i}{\partial r_j} s_j \right] = D^v_u U_i + D^s_u U_i. \]

Likewise, for the second equation of (2.37), one can show that

\[ \sum_{j=1}^{k} \frac{\partial V_i}{\partial r_j} s_j = \sum_{m=1}^{k} \sum_{n=1}^{k} \frac{\partial^2 U_i}{\partial r_n \partial u_m} v_m s_n + \sum_{m=1}^{k} \sum_{n=1}^{k} \frac{\partial^2 U_i}{\partial r_n \partial v_m} s_m v_n \]
\[ = \nabla_r (\nabla_u U_i \cdot v) \cdot s + \nabla_r (\nabla_r U_i \cdot s) \cdot s = D^{sr}_{uu} U_i + D^{sr}_{rr} U_i. \]

Similarly,

\[ \sum_{j=1}^{k} \frac{\partial V_i}{\partial u_j} v_j = D^{uv}_{uu} U_i + D^{uv}_{ru} U_i \]
\[ \sum_{j=1}^{k} \frac{\partial V_i}{\partial v_j} \dot{v}_j = D^{rv}_{ru} U_i \]
\[ \sum_{j=1}^{k} \frac{\partial V_i}{\partial s_j} \dot{s}_j = D^{rs}_{ru} U_i \]
\[ \sum_{j=1}^{k} \frac{\partial V_i}{\partial r_j} \dot{r}_j = D^{sv}_{ru} U_i \]

and therefore, for \( i = k + 1, \ldots, N \)
\[ D_{uu}^{vv} U_i + 2 D_{ru}^{sv} U_i + D_{rr}^{ss} U_i + D_u^{Fv} U_i + D_r^{Fs} U_i = g_i. \]  

(2.38)

In general, these differential equations are at least as difficult to solve as the original differential equations of motion, but they do allow for approximate solutions. Specifically, Theorem 2.4 can be used to reduce the size and the complexity of (2.38) and replace them with a much simpler algebraic equation. In this regard, Theorem 2.4, (2.20) states that solutions of (2.25) accept the form

\[ U_i = \sum_{n \in \mathbb{Z}} C_n^{(i)} e^{int}. \]

Therefore, when the equilibrium is not unstable and the condition defined by (2.18) holds for the linear system associated with the vector field \( X \) along the equilibrium, then one can simplify (2.38) since

\[ D_r^{sv} U_i = 0 \Rightarrow D_r^{sv} U_i = D_r^{ss} U_i = D_r^{Fs} U_i = 0. \]

Hence, the general constraint surface given by (2.25), which contains all solutions off the system (2.25) reduces to

\[ D_{uu}^{vv} U_i + D_u^{Fv} U_i = g_i, \ i = k + 1, \ldots, N, \]  

(2.39)

a \( 2k \)--dimensional invariant manifold that asymptotically contains only the marginally stable solutions of the system (2.25). This \( 2k \)--dimensional invariant manifold, which satisfies (2.38), defines the \( 2k \)--dimensional "Instantaneous Center Manifold" (ICM) for the nonlinear system (2.25). This manifold is tangent to the center subspace of the linear representation of the system at the equilibrium point of the nonlinear system and is the constraint surface on which asymptotically lie all stable solutions of the system.
It is important to mention, that ICM can be defined for both conservative and non-conservative systems. For a conservative system, the ICM simply represents the local center manifold of the system embedded in lower dimensions. For non-conservative (and some conservative) systems which don’t have (local) center manifolds, the ICM represents the periodic quotient of the (local) stable manifolds. The transformation which was used to transform the augmented differential EOM is the key to split the invariant manifolds of the system into non-overlapping periodic (ICM) and non-periodic (which may be called Instantaneous Stable Manifold-ISM) where non-overlapping quotients implies that their direct product reconstructs the stable manifolds.

Moreover, as mentioned earlier, Shaw and Pierre’s invariant manifold approach was restricted to only fixed points of the system. The main reason for such a limitation can be traced back to the fact that all these invariant manifolds are (and must be) defined locally. It means that one must represent these manifolds in a system of coordinates that is defined around the fixed point or periodic orbit of the system. In other words, the manifold $\Gamma$ should be formulated as $\Gamma(x - \bar{x}, y - \bar{y})$ where $(\bar{x}, \bar{y})$ represents the fixed point or periodic orbit of the system. In the case of fixed points, a simple translation $(z, w) = (x - \bar{x}, y - \bar{y})$ can be used to move the fixed point to the origin which will one to use virtually any form for the manifold $\Gamma(z, w)$ (Shaw and Pierre used a general Taylor series representation). However, when dealing with periodic orbits, formulating the manifold $\Gamma$ as $\Gamma(z, w)$ is far more difficult since it, first, requires finding the periodic solution, i.e. $(\bar{x}, \bar{y})$ which can be extremely challenging, and second a transformation from the system of coordinates $(x, y)$ to the one defined by $(z, w) = (x - \bar{x}(x, y), y - \bar{y}(x, y))$ which will not be a simple translation anymore. In this regard, the ICM representation can make performing the two mentioned steps as easy as equating $\dot{v} = F_v = 0$ in Eq. (2.39). That is because, defining $\dot{v} = 0$ not only restricts the solutions of (2.39) to only periodic solutions but also, makes any periodic solution the origin of the system of coordinates used in (2.39).
In this regard, Eq. (2.39) reduces to \(^{15}\)

\[
D^{uv} U_i = g_i, \ i = k + 1, \ldots, N.
\] (2.40)

Moreover, using the ICM formulation, one can define nonlinear modes for both conservative and non-conservative systems as eigensolutions of the system (2.39) (solutions of (2.40)) or solutions of the system that are both invariant and periodic. This not only can explain the relationship between the Vakakis nonlinear modes and Pierre and Shaw invariant manifolds (in conservative systems), it can also predict when the assumptions made either in their definitions or in methods of calculating nonlinear modes are not valid.

Furthermore, by precisely connecting the concept of nonlinear mode to center manifold theory, normal form theory and invariant manifolds of nonlinear dynamical system, the concept of ICM allows one to take full advantage of vast variety of well established theories and methods in nonlinear dynamics. For example, as shown in [7] and Chapter 5, one can use singularity and catastrophe theory to explain jump phenomena as a result of interaction between ICMs of the system in the state space.

\(^{15}\)This way, since all solutions of (2.40) are periodic, they accept a simplified version of the general form (2.20) which is equivalent to the conventional Fourier series expansions of the mentioned solutions. This property gave rise to one of the two algorithms for finding periodic orbits which will be discussed in the following Chapter.
Chapter 3

Calculation of Nonlinear Modes

In this section, ICM and Modes of vibration on the ICM are compared to Shaw and Pierre’s invariant manifolds and modified Rosenberg’s NNMs. Three examples representing linear conservative, linear non-conservative and nonlinear conservative cases with no special symmetry are presented respectively.

3.1 Finding Nonlinear Modes on a Two-Dimensional ICM

This approach provides a unique opportunity to interpret the system in Eq. (2.40). Its eigensolutions can be interpreted as modes of vibrations, i.e. $U_i$ as periodic (or periodic parts of) solutions of the system (2.25). In addition, the choice of independent variables, i.e. $(a, \phi)$, is perfectly suited to parametrize the periodic solutions or the periodic part of solutions (2.6). In this regard, it is shown in [11] that all solutions $U_i$ are in the form of

$$U_i = \sum_{m \in \mathbb{N}} \left[ A^{(i)}_m \cos(m\phi) + B^{(i)}_m \sin(m\phi) \right], \quad i = 2, ..., n.$$  \hspace{1cm} (3.1)

Therefore, (2.40) can be further simplified by substituting $U_i$ from Eq. (3.1) to yield
\[-\omega^2 \sum_{m \in \mathbb{N}} n^2 \left[ A_m^{(i)} \cos(m \phi) + B_m^{(i)} \sin(m \phi) \right] = g_i\]

where \(i = 2, ..., n\). Finding the eigensolution of the ICM defined by (2.40) is equivalent to finding the unknown coefficients in (3.2), i.e. \(A_m^{(i)}, B_m^{(i)}\). Defining the coordinate \(\phi\) and constants \(m\) such that \(\phi \in (-\pi, \pi]\) and \(m \in \mathbb{N}\) yields

\[
\begin{align*}
\int_{-\pi}^{\pi} \sin m \phi \cos n \phi d\phi &= 0 \\
\int_{-\pi}^{\pi} \cos m \phi \cos n \phi d\phi &= \pi \delta_{mn} \\
\int_{-\pi}^{\pi} \sin m \phi \sin n \phi d\phi &= \pi \delta_{mn}.
\end{align*}
\]

Then, multiplying (3.2) by \(\sin(mu)\) and \(\cos(mu)\) and integrating over the interval \((-\pi, \pi]\) yields

\[
\begin{align*}
-\pi v_1^2 m^2 A_m^{(j)} &= \beta^m, \quad j = 2, ..., n, \\
-\pi v_1^2 m^2 B_m^{(j)} &= \alpha^m.
\end{align*}
\]

where \(m \in \mathbb{N}\) and

\[
\begin{align*}
\alpha^m &= \int_{-\pi}^{\pi} g_i \sin m \phi d\phi \\
\beta^m &= \int_{-\pi}^{\pi} g_i \cos m \phi d\phi.
\end{align*}
\]

Equation (3.4) defines, in general, a nonlinear system of infinitely many algebraic equations with infinitely many unknowns, since \(m\) covers all natural numbers.

### 3.2 Analytical Examples

In order to calculate the ICMs, in general, one needs to perform three tasks. First to choose the independent coordinates. Second, to formulate the function \(g\) in (2.40) and third to form the algebraic system of equations (3.4). These tasks, considering the system (2.6), are outlined in more details below.
1. Find the off-diagonal element of the $J_f$ with the lowest cumulative order\(^1\). The lowest order, of
the element $\frac{\partial f_i}{\partial x_j}$, $j\neq i\neq j$, provides a criterion to select the pair of coordinates as independent
coordinates, i.e. $(x_{j\ast}, \dot{x}_{j\ast})$, that causes the lowest computational cost in the following steps.

2. Based on the function $f_{i\ast}$, found in step one, find $x_j$ as a function of the remaining coordinates,
i.e. $x_{j\ast} = h(\{x_1, \ldots, x_{j\ast-1}, x_{j\ast+1}, \ldots, x_n\})$.

3. Substitute all the dependent coordinates with $x_i = \sum_n [A_i^{(i)} \cos n\phi + B_i^{(i)} \sin n\phi]$, $i \neq j\ast$ (and
consequently $\dot{x}_i = \sum_n n\omega [B_i^{(i)} \cos n\phi - A_i^{(i)} \sin n\phi]$) where $\phi = \arg(H_{x_{j\ast}}, x_{j\ast})$ and $\omega = \frac{d}{dt} \phi$.

4. Find the function $x_{j\ast}(\phi) = h(\{x_1, \ldots, x_n\} - \{x_{j\ast}\})$ with the mentioned substitution in step 3.
Consequently substitute $x_{j\ast}$ by the function $x_{j\ast}(\phi)$ in all the functions in the force vector $f$.

5. Form the algebraic system of equations (3.4) (Note that all the functions in $f$ now are functions
of $\phi$).

This way, solving the algebraic system of equations (3.4) will reveal the periodic (or periodic parts
of) invariant solutions of the system (2.6) and then the relation between each coordinate and the
independent set of coordinates can be constructed, i.e. the invariant manifold of ICM. In the case of
linear systems, one can skip the steps 1, 2 and 4.

### 3.2.1 A Linear Conservative System

The linear conservative system to be considered is obtained from the system depicted in Fig. 2.2 by
setting $c_1 = c_2 = c_3 = 0$ and $k_{nl} = 0$. Therefore, the equations of motion (EOM) are

\(^1\)Assuming $\frac{\partial f_i}{\partial x_j} = \sum_k x_1^{(k)} \cdots x_n^{(k)}, i \neq j$ the cumulative order of the $ij$-th element is defined by $O_{ij} = \sum_k p_{ij}^{(k)}$. 


\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{y}_1 \\
\dot{y}_2
\end{pmatrix} =
\begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-(k_1 + k_2) & k_2 & 0 & 0 \\
k_2 & -(k_2 + k_3) & 0 & 0
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
y_1 \\
y_2
\end{pmatrix}.
\]

Assuming that \(x_1 = a_1 \cos u_1\) is a solution of the system (3.5), the function \(g_2(a, \phi)\), mentioned in (3.2) (after performing step 3), can be found as

\[
g_2(a, \phi) = f_2(x(a, \phi), y(a, \phi)) = k_2(a \cos \phi) - (k_2 + k_3)\sum_m[A_m^{(2)} \cos m\phi + B_m^{(2)} \sin m\phi].
\]

This way, the integral equations of (3.4) can be written as

\[
\begin{align*}
B_p^{(2)} &= -\frac{1}{\pi \omega^2 p^2} \int_{-\pi}^{\pi} \{k_2 a \cos \phi - (k_2 + k_3)\sum_m[A_m^{(2)} \cos m\phi + B_m^{(2)} \sin m\phi]\} \sin p\phi \, du_1 \\
A_p^{(2)} &= -\frac{1}{\pi \omega^2 p^2} \int_{-\pi}^{\pi} \{k_2 a \cos \phi - (k_2 + k_3)\sum_m[A_m^{(2)} \cos m\phi + B_m^{(2)} \sin m\phi]\} \cos p\phi \, du_1.
\end{align*}
\]

One can readily show that \(B_p^{(2)} = 0\) and

\[
A_p^{(2)} = -\frac{1}{\pi \omega^2 p^2} \left[\pi k_2 a \delta_{1p} - \pi (k_2 + k_3) A_p^{(2)}\right]
\]

which yields

\[
A_p^{(2)} = \frac{k_2}{-(k_2 + k_3) + \omega^2 a \delta_{1p}}.
\]

Therefore \(x_1 = a \cos \phi\) and \(x_2 = \frac{k_2}{-(k_2 + k_3) + \omega^2} a \cos \phi\) form the invariant periodic solutions of system (3.5). Note that \([x_1, x_2]^T = \left[1, \frac{k_2}{-(k_2 + k_3) + \omega^2}\right]^T x_1\) are the same as the LNMs of the system (3.5) if \(\omega\) is replaced by either of the natural frequencies of the system (3.5). However for the ICM approach, \(\omega\) is found by solving
\[ g_1 = -a\omega^2 \cos \phi, \]

which, since \( g_1 = -(k_1 + k_2)(a \cos \phi) + k_2 \frac{k_2}{-(k_2 + k_1) + v_1} a \cos \phi, \) leads to

\[ \omega^4 - [(k_1 + k_2) + (k_2 + k_3)]\omega^2 + [(k_1 + k_2)(k_2 + k_3) - k_2^2] = 0. \]

The above equation is the same as the characteristic equation of the eigenvalue problem for the system \((3.5) \ [110]\). Therefore, modes of vibration on ICMs of the linear system \((3.5)\) are the same as the LNMs of the system. Moreover ICMs defined by

\[ x_2 - \frac{k_2}{-(k_2 + k_3) + \omega^2} x_1 = 0 \]

are the same as the invariant manifolds of Shaw and Pierre for this system \([123]\).

### 3.2.2 A Linear Non-conservative System

The linear non-conservative system to be considered is obtained from the system in Fig. 2.2 by setting \( k_{nl} = 0 \). Therefore the EOM are

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{y}_1 \\
\dot{y}_2
\end{pmatrix} = -
\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
(k_1 + k_2) & -k_2 & (c_1 + c_2) & -c_2 \\
-k_2 & (k_2 + k_3) & -c_2 & (c_2 + c_3)
\end{bmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
y_1 \\
y_2
\end{pmatrix}.
\]

Assuming that \( x_1 = a \cos \phi \) is a solution of \((3.6)\), since

\[ f_2(x, y) = k_2x_1 + c_2y_1 - (k_2 + k_3)x_2 - (c_2 + c_3)y_2, \]

\( g_2(a, \omega) \) can be found by
\[ g_2(a, \phi) = k_2 a \cos \phi - c_2 \omega \sin \phi - (k_2 + k_3) \Sigma \left[ A_m^{(2)} \cos m \phi + B_m^{(2)} \sin m \phi \right] - \omega (c_2 + c_3) \Sigma \left[ -m A_m^{(2)} \sin m \phi + m B_m^{(2)} \cos m \phi \right]. \]

One can show that it leads to

\[
\begin{align*}
A_1^{(2)} &= \frac{c_2 \omega [\omega (c_2 + c_3)] - k_2 [-\omega^2 + (k_2 + k_3)]}{\omega (c_2 + c_3)^2 + [-\omega^2 + (k_2 + k_3)]^2} a_1, \\
B_1^{(2)} &= \frac{-k_2 [\omega (c_2 + c_3)] - c_2 \omega [-\omega^2 + (k_2 + k_3)]}{\omega (c_2 + c_3)^2 + [-\omega^2 + (k_2 + k_3)]^2} a_1. \\
\end{align*}
\] (3.7)

Note that in the case \( m \neq 1 \), \( A_m^{(2)} = B_m^{(2)} = 0 \) (trivial solution) since \( [\omega (c_2 + c_3)]^2 + [-\omega^2 + (k_2 + k_3)]^2 \neq 0 \). To this end, the solution, expressed in physical coordinates as

\[
\begin{align*}
x_2 &= \frac{c_2 \omega [\omega (c_2 + c_3)] - k_2 [-\omega^2 + (k_2 + k_3)]}{\omega (c_2 + c_3)^2 + [-\omega^2 + (k_2 + k_3)]^2} \left\{ \begin{array}{c} x_1 \\ \dot{x}_1 \end{array} \right\} = \frac{-k_2 [\omega (c_2 + c_3)] - c_2 \omega [-\omega^2 + (k_2 + k_3)]}{\omega (c_2 + c_3)^2 + [-\omega^2 + (k_2 + k_3)]^2} \left\{ \begin{array}{c} x_1 \\ \dot{x}_1 \end{array} \right\}, \\
\end{align*}
\] (3.8)

reveals the same relation as LNMs of the system if \( \omega \) takes the values of the damped natural frequencies of the system (3.6). It’s worth noting that the solutions (LNMs) depend on the velocity, whereas they do not for conservatives systems such as (3.5). This becomes evident without assuming this type of dependence when defining the ICMs. Finally, \( \omega \) can be obtained by solving

\[
\frac{c_2 (c_1 + c_2) \omega^2 + k_2 ((k_1 + k_2) - \omega^2)}{c_2^2 \omega^2 + k_2^2} = \frac{c_2 (c_2 + c_3) \omega^2 - k_2 [((k_2 + k_3) - \omega^2)]}{\omega (c_2 + c_3)^2 + [(k_2 + k_3) - \omega^2]^2},
\]

which is is the same as the characteristic equation of the eigenvalue problem for the system (3.6) [110].

This example illustrates how the ICM approach can be applied to non-conservative systems. Moreover, the ICMs represented in physical coordinates in (3.8) are the same as the invariant manifolds of Shaw and Pierre for this system [123]. However this is only the case for linear non-conservative systems. That is because, as evidenced in (3.7), the aperiodic part, i.e. \( a \), is the same for all coordinates defining the ICMs. In nonlinear non-conservative systems, since the aperiodic part will be different for each coordinate.
coordinate, the ICMs will have an entirely different representation than the invariant manifolds of Shaw and Pierre.

3.2.3 A Nonlinear Conservative System

The nonlinear conservative system can be obtained from the system in Fig. 2.2 by setting $c_1 = c_2 = c_3 = 0$. In this case, EOM are

$$
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{y}_1 \\
\dot{y}_2
\end{pmatrix} = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-(k_1 + k_2) & k_2 & 0 & 0 \\
k_2 & -(k_2 + k_3) & 0 & 0
\end{pmatrix} \begin{pmatrix}
x_1 \\
x_2 \\
y_1 \\
y_2
\end{pmatrix} + \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-k_n x_1^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \begin{pmatrix}
x_1 \\
x_2 \\
y_1 \\
y_2
\end{pmatrix}.
$$

(3.9)

Suppose that $x_1 = a \cos \phi$ is a periodic solution of the system (3.9). Note that $x_1$ is not constrained to be harmonic, i.e. $\dot{a}_1 \neq 0$ in general. Step 1 is performed and the function $f_2$ and the coordinate $x_1$ are selected\(^2\). Since

$$f_2(x, y) = k_2 x_1 - (k_2 + k_3) x_2$$

then, the function $g_2(a, \phi)$ can be obtained as

$$g_2(a, \phi) = k_2 (a \cos \phi) - (k_2 + k_3) \sum \left[ A_n^{(2)} \cos m\phi + B_n^{(2)} \sin m\phi \right].$$

This way, $x_1$ is obtained as

$$x_1 = a \cos \phi = \frac{1}{k_2} \sum \left( k_2 + k_3 - m^2 v^2 \right) \sum \left[ A_n^{(2)} \cos m\phi + B_n^{(2)} \sin m\phi \right] = \frac{1}{k_2} \sum \alpha_m \left[ A_n^{(2)} \cos m\phi + B_n^{(2)} \sin m\phi \right].$$

\(^2\)Commutative order of $\frac{\partial f_2}{\partial x_1}$ is zero.
where $\alpha_m = (k_2 + k_3 - m^2 \omega^2)$ (step 2).

From (3.9) $g_1$ can be defined as

$$g_1(a, \phi) = \frac{d^2}{dt^2} (a \cos \phi) = -a(k_1 + k_2) \cos \phi + k_2 \sum_m \left[ A_m^{(2)} \cos m\phi + B_m^{(2)} \sin m\phi \right] - k_{nl} \{a \cos \phi\}^3.$$  

Substituting $x_1$ and $x_2$ in $g_1$ yields

$$\sum \left[ \gamma_m A_m^{(2)} \cos m\phi + \gamma_m B_m^{(2)} \sin m\phi \right] = \frac{k_{nl}}{k_2^2} \left\{ \sum \left[ \alpha_m A_m^{(2)} \cos m\phi + \alpha_m B_m^{(2)} \sin m\phi \right] \right\}^3$$

where $\gamma_m = k_3^2 - (k_1 + k_2 - m^2 \omega^2) \alpha_m$ (step 3). $A_i = A_i^{(2)}$ and $B_i = B_i^{(2)}$, $i \in \mathbb{N}$, are the solutions of the nonlinear system of algebraic equations (steps 4-5)

$$\gamma_i A_i = \frac{k_{nl}}{k_2^2} \int_{-\pi}^{\pi} \left\{ \sum \left[ \alpha_m A_m^{(2)} \cos m\phi + \alpha_m B_m^{(2)} \sin m\phi \right] \right\}^3 \cos i u \ du$$

$$\gamma_i B_i = \frac{k_{nl}}{k_2^2} \int_{-\pi}^{\pi} \left\{ \sum \left[ \alpha_m A_m^{(2)} \cos m\phi + \alpha_m B_m^{(2)} \sin m\phi \right] \right\}^3 \sin i u \ du.$$  

Solving (3.10) will determine $x_2$ and consequently $x_1$ as functions of instantaneous frequency $\omega$. For instance, the nonlinear algebraic equations corresponding to the approximation of the system in (3.10), up to 4-th order harmonics, is provided here.

$$\gamma_1 A_1 = \frac{3k_{nl}}{4k_2^2} \left[ A_1^2 \alpha_1^2 + A_2^2 A_3^2 \alpha_3 + 2A_2 A_3 \alpha_2 A_3 (A_2 \alpha_2 + A_3 A_4) + 2A_1 \alpha_1 (A_2^2 \alpha_2^2 + A_3^2 \alpha_3^2 + A_2 A_4 \alpha_2 \alpha_4 + A_3^2 \alpha_3^2) \right]$$

$$\gamma_2 A_2 = \frac{3k_{nl}}{4k_2^2} \left[ A_2 \alpha_2 (2A_1^2 \alpha_1^2 + A_2^2 A_3^2 + 2A_1 A_3 A_2 \alpha_3 + 2A_3^2 \alpha_3) + 2A_1 \alpha_1 (A_2^2 \alpha_2^2 + A_3^2 \alpha_3^2 + A_2 A_4 \alpha_2 \alpha_4 + A_3^2 \alpha_3^2) \right]$$

$$\gamma_3 A_3 = \frac{3k_{nl}}{4k_2^2} \left[ A_3^2 \alpha_3^2 + 6A_2^2 A_3 \alpha_2 \alpha_3 + 3A_1 A_2 \alpha_2 \alpha_2 (A_2 \alpha_2 + A_3 \alpha_3) + 2A_1 \alpha_1 (A_2^2 \alpha_2^2 + A_3^2 \alpha_3^2 + 2A_2 A_4 \alpha_2 \alpha_4 + 2A_3^2 \alpha_3^2) \right]$$

$$\gamma_4 A_4 = \frac{3k_{nl}}{4k_2^2} \left[ A_2 \alpha_2 (A_1 \alpha_1 + A_3 \alpha_3)^2 + 2A_4 (A_2^2 \alpha_2^2 + A_3^2 \alpha_3^2 + A_3^2 \alpha_3^2) \right].$$

### 3.2.4 Solving a Nonlinear Eigenvalue Problem

Equation (3.4), in nonlinear cases, always leads to an under-determined system of algebraic equations because of the extra unknown variable $\omega$. In order to overcome this shortcoming, one has to either add an extra equation (constraint) or start from a known solution. These approaches are explained in two schemes,namely continuation and sweeping schemes.
3.2.4.1 Continuation Scheme

Assuming that all the unknowns are continuous functions, one starts from a known solution (state) and, by changing the state slightly, searches for a new solution that satisfies the system of equations. The known solutions are usually the natural frequencies of the underlying linear system and its LNMs at a very low energy where nonlinearity has a negligible effect.

For example, to solve (3.11), one can start at the initial state \( s \) as

\[
s = [\omega, A_1, A_2, A_3, A_4]^T
\]

where \( \omega \) is the natural frequency of the system and \( A_i, i = 1, \ldots, 4 \) are the coefficients of the harmonic terms in the approximation of the corresponding LNM. Then, for the next solution, \( \omega \) will be replaced by a prediction \( \omega_p = \omega + \Delta \omega \) where \( \Delta \omega \) is a small real constant and the Eq. (3.11) will be solved for its companion set of coefficients \( A_i, i = 1, \ldots, 4 \). This same scheme is used in a well-established approach, which is discussed in [105], although that scheme uses numerical integration and physical coordinates. The periodic motions found utilizing the scheme in [105] are shown in (3.1). These were found to match the solution using the continuation algorithm discussed above very precisely at all the states that were found by that approach.
Figure 3.1: Two main branches of nonlinear modes obtained by continuation of LNMs. The region revealing cases of internal resonance is magnified. It is worth mentioning that the algorithm based on the continuation of the parameter $\omega$, needs to be reinitialized wherever the function $\frac{\partial \omega}{\partial E}$, in which $E$ is the energy of the system, changes sign.

Figure 3.2: (a): Displacement invariant manifold calculated from the first main branch of NNMs. (b): The middle section of the manifold shows the overlaid Shaw and Pierre invariant manifold. (c): Velocity invariant manifold calculated from first mode branch of nonlinear modes. (d): The middle section of the manifold shows the overlaid Shaw and Pierre invariant manifold.

It is important to notice that the ICM approach does not require one to recruit different methods for internal resonance cases unlike the Shaw and Pierre’s invariant manifold approach (see [75]). Moreover, the ICM method can benefit directly from NNMs, i.e. periodic solutions, found using the Modified Rosenberg approach (or any other numerical method) to construct the ICMs and consequently, in the case of conservative systems, the Shaw and Pierre invariant manifolds. This allows one to utilize Rosenberg’s NNMs, which up until now were mostly used only to characterize the system’s behavior.
qualitatively, to predict the response of the system quantitatively using either the ICMs or the resultant Shaw and Pierre invariant manifolds. In this sense, Figs. 3.2 and 3.3 represent the Shaw and Pierre’s invariant manifolds obtained from ICMs calculated using the nonlinear modes found in Fig. 3.1. As shown in Fig. 3.2, the invariant manifold, obtained from the corresponding ICM, not only contains the Shaw and Pierre’s invariant manifold but all the internal resonance case shown in Fig. 3.1 (right). Figure 3.3, also reveals a complex structure at higher energies which had been overlooked in Shaw and Pierre’s invariant manifolds approximation.

Figure 3.3: (a): Displacement invariant manifold calculated from the second main branch of nonlinear modes. (b): The middle section, representing the higher energy modes, reveals a double-peak shape manifold that can not be seen in the Shaw and Pierre’s invariant manifold (with up to 3rd order approximations). (c): Velocity invariant manifold calculated from second mode branch of nonlinear modes. (d): The middle section shows the overlaid Shaw and Pierre’s invariant manifold.

3.2.4.2 Sweeping Scheme

The core idea in this scheme is to either parametrize one of the unknowns or to add an extra constraint to balance the the system of equations (3.4). The former can be explained as that the system (3.4) will be solved for all feasible solutions at each given value of one selected unknown. Then, the mentioned unknown will sweep an entire range of interest. In the latter, one parametrizes an extra constraint instead of an unknown.
- Frequency Sweeping: The frequency of the periodic motions is parametrized. As illustrated in Fig. 3.4, in this scheme, one searches for all periodic motions with the same (fundamental) frequency that can sweep any range of interest.

- Energy Sweeping: The extra constraint is achieved by parametrization of the energy of the system at any given periodic motion. This way, as illustrated in Fig. 3.4, one searches for all periodic motions along the constant energy line that will sweep an entire range of interest.

This scheme provides a unique opportunity to search for all the existing periodic motions. As mentioned in the Section 1, for each fixed level of energy there exist at least \( n \) periodic solutions (modes) passing through each stable equilibrium which leaves the possibility of the existence of extra branches of nonlinear modes open. Utilizing the mentioned sweeping scheme revealed new branches of nonlinear modes that had been remaining latent using the continuation scheme in the modified Rosenberg approaches. Figure 3.4 shows some of these extra branches.

Figure 3.4: Left: branches of nonlinear mode found by the ICM approach and sweeping scheme. Equation (3.10) can be solved along either the constant frequency or the constant energy constraint line. Right-Top: Same complex curves of solutions can be found using the sweeping scheme. Right-Bottom: Solving Eq. (3.10) along the constant energy line reveals solutions that do not belong to the two main branches (which are a continuation of the LNMs). Many more of these branches exist than are shown here.
The relationship between the new branches, which are shown in Fig. 3.4, and the two main branches (continuation of LNMs) are depicted in Fig. 3.5. In this regard, the main two branches are repeated for all integer multiples of their period, since, if $T$ is the fundamental period of a nonlinear mode so is all $T_n = nT$, $n \in \mathbb{N}$. Note that the reverse argument is not valid, i.e. $T/n$ is not a fundamental period.

In other words, visually, any branch of nonlinear modes can be pushed down, and only down, in the Energy-Frequency plane to frequencies $\omega_n = \omega/n$, $n \in \mathbb{N}$. These branches are indicated by a subscript, representing the mode number that the original branch stands for, and a superscript representing the decremental factor $1/n$ of the frequencies of the original branch. For instance, $M_1^{1/4}$ represent the branch of first nonlinear modes with frequencies that are $\omega_1^{1/4} = (1/4)\omega_1$. After doing this, it becomes clear that the new branches reveal four different shapes marked by letters (a), (b), (c) and (d) in Fig. 3.5. The first three types, i.e. (a), (b) and (c), connect the first and second main branches and are different only in shape and size. The last type, i.e. (d), travels between two nonlinear modes of the same main branch.

Figures 3.6 and 3.7 shed light on various aspects of these new branches. First, since end points of these branches represent modes from the main branches, they, in a sense, connect the main branches of nonlinear modes. Second, any point in between endpoints represent modes that do not resemble any nonlinear modes in their neighborhood from the main branches. Third, these new branches contain both in-phase and out-of-the-phase modes even in the case of connecting two modes of the same (all in-phase nonlinear modes) branch. Finally, the time domain representation of theses intermediate modes show that these modes do not replicate any of nonlinear modes on the main branches since their fundamental periods (frequency) are much larger (smaller) than the ones from the nonlinear modes of the main branches.
Figure 3.5: Some of the numerous branches of nonlinear modes that can not be obtained by continuation of LNMs shown versus frequency and Energy. The main two branches of nonlinear modes are also shown with many of their possible periods indicated by $M_j^{1/i}$. The subscript represents the mode number of the original branch and the superscript represents the decremental factor $1/n$ of the frequencies of the original branch, e.g. $M_1^{1/4}$ represent the branch of first nonlinear modes with frequencies that are $\omega_1^{1/4} = (1/4)\omega_1$.

Figure 3.6: Top: The solid closed curve represents the state variable $x_1$ while the dashed one represents the state variable $x_2$ ($\dot{x}_1 = \dot{x}_2 = 0$) associated with branch (a) in Fig. 3.5. This, forming a closed curve, implies that the branch (a) in Fig. 3.5 may be an isolated branch. Middle: Four modes, corresponding to the four highlighted sets of initial conditions (a), (b), (c) and (d) in the upper plot, are represented. Modes shown in (a) and (b) are in-phase and modes (c) and (d) are out-of-the-phase. Note that (b) and (d) suggest that this branch starts from second branch and travels to the first one.
Figure 3.7: Left: The solid closed curve represents the state variable $x_1$ while the dashed one represents the state variable $x_2$ ($\dot{x}_1 = \dot{x}_2 = 0$) associated with branch (d) in Fig. 3.5. Middle: Branch (d) in Fig. 3.5. Right: Four modes, corresponding to the four highlighted sets of initial conditions (a), (b), (c) and (d) in the upper plot, are represented. Mode shown in (a) is in-phase and modes (b),(c) and (d) are out-of-the-phase. Note that (b) and (d) suggest that this branch separates from second branch and joins back to it.

Figure 3.8: Top: Time history of two modes, marked as point (a) and (c) in Fig. 3.6 are represented. Although they have very similar mode shapes, one is in-phase and the other one is out-of-the-phase. Bottom: Time history of two modes, marked as points (a) and (c) in Fig. 3.7 are represented. One is in-phase and the other one is out-of-the-phase.

3.3 Finding Nonlinear Modes of Larger Structures

Finding periodic orbits is often the first step in studying bifurcation or chaos in nonlinear systems and also in determining frequency response functions (or frequency-energy characteristic curves) of such systems. To find periodic orbits, it is important to first explain how one can prove that a solution of a
system of (autonomous) differential equations is periodic. According to the uniqueness theorem [21], solutions of autonomous differential equations do not cross each other or themselves. If this were not the case, there would exist a point in the state space such as $y_0$ in Fig. 3.9(b) that would initiate two different solutions. This implies that if a solution of a (set of) differential equation(s) passes through the same point $y_0$ at two distinct times $t_1$ and $t_2$, it must be a periodic solution with a period of $T = t_2 - t_1$.

Figure 3.9: Periodic solutions are the only solutions of autonomous differential equation that may pass through a point in the state space more than once over time.

This relationship is the fundamental idea shared in all algorithms that attempt to find periodic orbits of autonomous differential equations. These algorithms can be divided into two categories based on how they use this concept. The first class of algorithms use the fact that all periodic functions have closed form approximations such as a Fourier series expansion. These algorithms attempt to fit one of these approximations to a solution of the system and then extract the set $\{y_0, T\}$ from the approximation form. For example, perturbation theory formulates the solution as a series expansion in small parameters [99], but unfortunately this limits the method to weakly nonlinear systems. Averaging methods, such as Harmonic-Balance, rely on a Fourier series expansion to approximate the periodic orbit of the system by averaging (evaluating an integral of the response or the vector field) it over one period. Variational methods can be used for strongly nonlinear systems, however, like perturbation and averaging methods, they rely on analytic procedures to find periodic orbits and are not readily extendable to large systems [26].
A second class of methods finds a set of an initial conditions and a period, i.e. \( \{ y_0, T \} \), for which, the solution of the system that is initiated at the initial condition \( y_0 \) returns to it after the time \( T \). In this sense, a different class of approaches, generally known as multi-point boundary value methods [113], restates the problem of finding periodic orbits of the system

\[
\ddot{x} = f(x), \ x \in \mathbb{R}^n. \tag{3.12}
\]

as finding a point \( y_0 \) for which

\[
y_0 = y(t_0) = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}_{t=t_0} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}_{t=t_0+T} = y(t_0 + T). \tag{3.13}
\]

Algorithms based on this approach are known as shooting (or multiple shooting in general) algorithms. In the case of multiple shooting, the interval \([t_0, t_0 + T]\) is divided into \( M \) sub-intervals \([t_p, t_{p+1}]\) and consequently the system (3.12) and the condition (3.13) and are

\[
\ddot{x}_p = f_p(x_p), \ p = 1, ..., M \tag{3.14}
\]

and

\[
\begin{bmatrix} x \\ \dot{x} \end{bmatrix}^{(p)}_{\tau=0} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}^{(p+1)}_{\tau=1}, \ p = 1, ..., M, \ p \geq 1 \tag{3.15}
\]

where \( x_p(\tau) = x(t_p + (t_{p+1} - t_p)\tau), \ \tau = \frac{t-t_p}{t_{p+1} - t_p} \). In other words, the two points in (3.13) are augmented by \((M - 2)\) intermediate points to help improve the convergence of the solution of the system to a periodic orbit. It is worth pointing out that, since the boundary points are found by integrating the vector field (always) initiated at \( y(t_0) \), these approaches implicitly impose the constraint that all the points in (3.13) (or (3.15)) must be on the same solution (see Fig. 3.10). Shooting algorithms are widely used to satisfy the mentioned conditions (3.13) or (3.15) by minimizing the shooting function, i.e.
These algorithms are usually paired with the principle of continuation \[2, 86\] to find a branch of solutions. For example, the widely used software AUTO \[34\] and MatCont \[41\] both use this approach. While these methods have proven effective in many scenarios, they are highly sensitive to initial values and, as averaging methods require evaluating the integral of the response over one period, shooting methods require that one integrate the equations of motion to find the time response over one period.

3.4 Multi-harmonic Multiple-point Collocation: a Periodicity Condition

In this section we propose an alternative periodicity condition that for special cases of monotonic vector fields is equivalent to the traditional periodicity condition for solutions of second order differential equations shown in Eq.(3.13). The proposed condition has been devised specifically to alleviate three limitations of the periodicity condition 3.13. First, it is represented in acceleration form. Second, it is formulated in terms of an analytical Fourier series form and third it can contain and search for more than one periodicity condition at once. In this regard, suppose that periodic solution of the system in (3.12) accept the Fourier series expansion of the form

\[
\bar{x}(t) = \sum_{k=1}^{N} \left\{ \begin{array}{c} A_{1k} \\
... \\
A_{nk} \\
\end{array} \right\} \cos(k\omega t) + \left\{ \begin{array}{c} B_{1k} \\
... \\
B_{nk} \\
\end{array} \right\} \sin(k\omega t) , \quad \omega = \frac{2\pi}{T}, \ T \in \mathbb{R}^+ , \tag{3.16}
\]
then the condition, here dubbed Multi-Harmonic Multiple-point Collocation (MMC), and expressed by

\[ \ddot{x}(\phi_k) - f(\dot{x}(\phi_k)) = 0, \quad k = 1, \ldots, 2M \]

\[ \phi_{i+M} = 2\pi + \phi_i, \quad i = 1, \ldots, M, \quad \phi_i \in [0, 2\pi] \]  

combines the strength of analytical approaches, i.e. analytical candidate solution (host) \( \bar{x} \), and the convenience of numerical methods that can be applied to larger systems with strong nonlinearity. The solution host can be differentiated analytically, so this condition can be checked by simply evaluating the forcing function \( f \). This way, by introducing the MMC condition as an alternative to the conditions (3.13) or (3.15), avoids integration of the vector field over any period of time. Two iterative algorithms, discussed in Sections 3.5.1 and 3.5.2, are used to adjust the coefficients, \( A_{nk} \) and \( B_{nk} \), and the period, \( T \), until the MMC condition is satisfied. The proposed method is exemplified for a simple two dimensional (\( n=2 \)) and a rather large ten dimensional (\( n=10 \)) system in Sections 3.6 and 3.7.

However, since Eq. 3.15 is the (one and) only condition that defines and consequently can guarantee periodicity of solutions of a homogenous, second order ordinary differential equation, it is helpful to elaborate on how the simple MMC condition can guarantee that the solutions found satisfy the aforementioned periodicity condition (3.15). For example, since one is not required to integrate from one point to another for each pair of collocation points in the MMC condition, each of the \( M \) pairs of points on the solution \( \bar{x} \) may come from a different solution of the system. Therefore, the \( \bar{x} \) that one finds is frequently not a solution of the system but what shall be called a solution host since it matches with different solutions of the system at each pair of collocation points. These concepts are discussed in detail in the following sections.
3.4.1 Multiple-point Collocation Condition

In this section, we extend the periodicity condition \(3.15\) to include multiple pairs of collocation points using a periodic candidate solution host \(\bar{y} = \begin{bmatrix} \bar{x} \\ \hat{x} \end{bmatrix}\). In this regard, first, we define four sets, each consisting of \(M\) points. These sets are \(\{y_i = y(\frac{\phi_i}{2\pi} T)\}\) and \(\{y_{i+M} = y(\frac{2\pi + \phi_i}{2\pi} T)\}\) in the state space accompanied by another two sets of \(M\) points \(\{\bar{y}_i = \bar{y}(\phi_i)\}\) and \(\{\bar{y}_{i+M} = \bar{y}(2\pi + \phi_i)\}\) on the candidate solution \(\bar{y}\) where \(\phi_i \in [0, 2\pi], i = 1, ..., M\). Then, we define a difference function

\[
d_k = \bar{y}(\phi_k) - y(\frac{\phi_k}{2\pi} T), k = 1, ..., 2M, \phi_{i+M} = 2\pi + \phi_i, i = 1, ..., M. \tag{3.18}\]

Now we propose a multiple collocation condition defined as

\[
d_k(\bar{y}, y) = 0, k = 1, ...2M. \tag{3.19}\]

One can view the condition \(3.19\) as a set of \(M\) conditions of the kind presented in Eq.(3.13).

Equation \(3.19\) not only infers that the \(2M\) points \(\bar{y}(\phi_k)\) and \(\bar{y}(2\pi + \phi_k)\) from the candidate solution collocate with the \(2M\) points \(y(\frac{\phi_k}{2\pi} T)\) and \(y(\frac{2\pi + \phi_k}{2\pi} T)\) on solution(s) of the system, but since by definition \(\bar{y}(\phi_k) = \bar{y}(2\pi + \phi_k)\), then the solution \(y\) crosses itself in the state space, i.e. \(y(\frac{\phi_k}{2\pi} T) = y(\frac{2\pi + \phi_k}{2\pi} T)\) resulting in a set of \(M\) pairs of collocated points on the actual solution(s) of the system. Hence, it is clear that if this condition \(3.19\) is satisfied, one has identified up to \(M\) distinct periodic solutions of the system \(3.12\).

However, it is important to notice that the conditions \(3.19\) and \(3.15\) are fundamentally different, since without integration, there is no constraint imposed on the \(M\) pairs of points to force them to be on the same solution of the system \(3.12\). In other words, the candidate solution \(\bar{y}\) can either identify a periodic solution of the system (if all collocation pairs of points converge on the same periodic solution \(y\) of the system) or a set of periodic solutions with which it shares only few (more than one collocation pair...
of points and less than \( M \) points without matching any single solution \( y \) entirely. In this latter case we call it a solution host. In this thesis, we call the former case of convergence homogenous convergence and the latter heterogeneous convergence. Figure 3.10 compares this condition with two-point and multiple-point collocation conditions used in the shooting and multiple shooting algorithms respectively.

### 3.4.2 Acceleration Form of Multiple-point Collocation Condition

We mentioned earlier that in the proposed method, one is not required to integrate the system over any period of time. However, to use the multiple-point collocation condition (3.19) would require integrating the vector field to find the state of the system \( y \). To resolve this conflict, instead of the condition (3.19), we consider

\[
\dot{d}_k(\vec{y}, y) = 0, \ k = 1, ..., 2M
\]

(3.20)

and explain how it leads to a new condition, which (for a broad class of \( f \)) is equivalent to multiple-point collocation condition in (3.19), but does not require one to integrate the vector field. Condition (3.20) requires that \( \dot{d}_k = 0 \), so the following conditions must be met if Eq. (3.20) is satisfied.

\[
\begin{align*}
\dot{x}(\phi_k) - \dot{x}\left(\frac{\phi_k}{2\pi}T\right) &= 0 \\
\ddot{x}(\phi_k) - \ddot{x}\left(\frac{\phi_k}{2\pi}T\right) &= 0 \\
\dot{x}(2\pi + \phi_k) - \dot{x}\left(\frac{2\pi + \phi_k}{2\pi}T\right) &= 0 \\
\ddot{x}(2\pi + \phi_k) - \ddot{x}\left(\frac{2\pi + \phi_k}{2\pi}T\right) &= 0 \\
&\quad k = 1, ..., M.
\end{align*}
\]

(3.21)

Since \( \dot{x} \) (by definition) is always periodic, then \( \dot{x}(\phi_k) = \dot{x}(2\pi + \phi_k) \) which, subtracting two of the equations in (3.21), implies that \( \ddot{x}\left(\frac{\phi_k}{2\pi}T\right) = \ddot{x}\left(\frac{2\pi + \phi_k}{2\pi}T\right) \) and consequently,
Figure 3.10: (a) Illustration of two-point collocation condition used in shooting algorithms. (b) Multiple-point collocation condition used in multiple shooting algorithms. (c) Multiple-point collocation condition used in the proposed algorithm. The 2M points are used to create M independent two-point collocation conditions.

\[
f(x^\phi_k T) = \ddot{x}(\phi_k T) = \ddot{x}(\frac{2\pi + \phi_k}{2\pi} T) = f(x(\frac{2\pi + \phi_k}{2\pi} T)). \tag{3.22}
\]

Therefore, if the vector function \( f \) which defines the vector field in Eq. (3.22) is monotonic, then Eq. (3.22) implies that \( x(\frac{\phi_k}{2\pi} T) = x(\frac{2\pi + \phi_k}{2\pi} T) \). Moreover, note that subject to the assumption of monotonicity of \( f \), the velocity part in Eq. (3.21) is also implied by the acceleration part and not required anymore. That is because, \( \ddot{x}(\frac{\phi_k}{2\pi} T) = \ddot{x}(\frac{2\pi + \phi_k}{2\pi} T) \) implies that \( \dot{x}(t_k) = \dot{x}(t_k + T) + c_k, c_k \in \mathbb{R} \), however, \( x(t_k + T) - x(t_k) = \int_{t_k}^{t_k+T} (\int_{t_k}^{t_k+T} \ddot{x} \, d\tau) \, dt = c_k T = 0 \) or equivalently \( c_k = 0 \). Therefore, for a monotonic vector field \( f \), Eq.(3.20) not only is equivalent to the multiple-point collocation condition in Eq. (3.19) but also reduces to
\[
\delta_k(x, x) = \ddot{x}(\phi_k) - f(x(\phi_k/2\pi)) = 0, \; k = 1, ..., 2M, \tag{3.23}
\]

which we refer to as the acceleration form of the multiple-point collocation condition.

### 3.4.3 Multi-harmonic Approximation of Acceleration Form of Multiple-point Collocation Condition

So far, both the multiple-point collocation condition (3.19) and its equivalent reduced form (3.23) are formulated using both the multi-harmonic solution host \(\bar{x}\) (or \(\bar{y}\)) and the yet unknown solution of the system \(x\) (or \(y\)). This section explains how this can be replaced with the alternate condition in Eq. (3.17), which replaces \(x\) with the solution host \(\bar{x}\).

In this regard, suppose that a (yet unknown) solution of the system \(x\) and a multi-harmonic candidate solution \(\bar{x}\) satisfy Eq. (3.23) (which makes \(x\) a periodic solution) but \(x(\phi_k/2\pi) = \bar{x}(\phi_k) + \delta_k\) where \(\delta_k \neq 0\) is small. This way, one can write

\[
\ddot{\bar{x}}(\phi_k) - f(\bar{x}(\phi_k)) = 0, \tag{3.24}
\]

or approximately

\[
\ddot{\bar{x}}(\phi_k) - f(\bar{x}(\phi_k)) - \mathcal{J}_f(\bar{x}(\phi_k))\delta_k \approx 0, \tag{3.25}
\]

which leads to

\[
\delta_k \approx \left[\mathcal{J}_f(\bar{x}(\phi_k))\right]^{-1} [\ddot{\bar{x}}(\phi_k) - f(\bar{x}(\phi_k))] = \left[\mathcal{J}_f(\bar{x}(\phi_k))\right]^{-1} \ddot{\bar{x}}(\bar{x}) \tag{3.26}
\]

Hence, if \(\mathcal{J}_f(\bar{x}(\phi_k))\) is not singular, one can conclude that the multi-harmonic multiple-point collocation condition, i.e.
\( \delta_k(\bar{x}) = \dot{x}(\phi_k) - f(\bar{x}(\phi_k)) = 0, \ k = 1, \ldots, 2M. \) \hfill (3.27)

guarantees that the points on the candidate solution host are on the solutions of the system that satisfy (3.23), or equivalently, \( \bar{x}(\phi_k) \) collocates with points on a periodic solution of the system, i.e \( x(\frac{\phi_k}{2\pi} T) \). In other words, the condition (3.27) is not only a first order approximation to the multiple-point periodicity condition (3.23) (and equivalently (3.19) ) but it also imposes a constraint that plays the same role as integration in multi-point boundary value problem methods by implicitly forcing the candidate solution \( \bar{x} \) to collocate with the solution of the system \( x \) at finite number of points in the state space.

### 3.5 Multi-harmonic Multiple-point Collocation: Numerical Algorithms

In this section we propose two algorithms to modify the candidate host \( \bar{x} \) until it satisfies the multi-harmonic multiple-point collocation condition (3.27).

#### 3.5.1 Inexact Golden-Section Line Search MMC

Here we propose a steepest descent algorithm augmented by an inexact golden-section line search subroutine. The mentioned inexact line search is performed to find the step-size at each iteration that satisfies Wolfe conditions (to find inexact optimal step-size at each iteration). The immediate and main advantages of using this algorithm is to first to relax the need to calculate the inverse of any slope functions (Jacobian, or Hessian), which is required in Newtonian algorithms, while providing the same level of accuracy in choosing a step-size as is usually associated with Newtonian methods. In this regard we first define a deviation function
\[ D = \sum_{i=1}^{2M} \delta^T \delta = \sum_{i=1}^{2M} [f(\bar{x}(\phi_i)) - \ddot{\bar{x}}(\phi_i)]^T [f(\bar{x}(\phi_i)) - \ddot{\bar{x}}(\phi_i)] \]  

(3.28)

Next, starting from an initial guess \( \bar{x}^{(0)} \) determined by \( \{C^{(0)}, \omega^{(0)}\} \) where

\[ C^{(0)} = [A_{11} \cdots A_{1N} | B_{11} \cdots B_{1N}] \cdots [A_{n1} \cdots A_{nN} | B_{n1} \cdots B_{nN}]^T \]  

(3.29)

and \( \omega^{(0)} \) is initial guess for the frequency of the multi-harmonic candidate host. We then find the next approximation by navigating the candidate solution \( \bar{x} \) in the directions which are functions of the gradient of \( D \) with respect to the coefficient vector \( C \), and the frequency \( \omega \) denoted by \( g_D^C \) and \( g_D^\omega \) respectively. Specifically,

\[ C^{q+1} = C^q - r_c v_C^q \]  
\[ \omega^{q+1} = \omega^q - r_\omega v_\omega^q. \]  

(3.30)

where \( v_C^k \) and \( v_\omega^k \) are search directions (defined below) and \( r_c \) and \( r_\omega = \alpha r_c, \alpha \in \mathbb{R}^+ \) are the step size control parameters, which are determined using an inexact Golden-Section line search method. The search directions, for both coefficient vector and frequency, are the steepest descent directions at the first step. Subsequent steps, however, are defined recursively by

\[ v_{C,k+1}^{k+1} = g_D^{C,k+1} + \text{diag}(\gamma_k^k) v_{C,k}^k, \quad v_0^{C,k} = g_D^{C,0} \]  
\[ v_{\omega,k+1}^{k+1} = g_D^{\omega,k+1} + \gamma_k^k v_{\omega,k}^k, \quad v_0^{\omega,k} = g_D^{\omega,0}, \]  

(3.31)

where \( \gamma^k \in \mathbb{R} \) is chosen as

\[ \gamma_{C,k}^k = \frac{\|g_D^{C,k+1}\|^2}{(v_{C,k}^k)^T (g_D^{C,k+1} - g_D^{C,k})} \]  
\[ \gamma_{\omega,k}^k = \frac{(v_{\omega,k}^k)^T (g_D^{\omega,k+1} - g_D^{\omega,k})}{(g_D^{\omega,k+1})^2} \]  

(3.32)

This reduces minimization of (3.28) to a conjugate gradient method minimization problem[53, 54, 52].

Specifically, at each iteration, a minimization problem defined by
\[
\min_{r_k^c} D(C^{q+1}, \omega^{q+1})
\]  
(3.33)

must be solved to find an optimal step-size \( r_k^c \). However, while an exact solution to Eq. (3.33) is not available, an inexact solution can be found by finding a step-size \( r_k^c \) using a Golden-section line search described by

\[
r_{c}^{k+1} = \frac{r_{c}^{k}}{\alpha_{GS}}, \quad r_{c}^{0} = r_{c_{max}}^{c} \in \mathbb{R}^{+},
\]  
(3.34)

where \( \alpha_{GS} = \frac{1+\sqrt{5}}{2} \) is the golden section ratio, that satisfies the (strong) Wolfe conditions

1. \( D(C^{q+1}, \omega^{q+1}) - D(C^{q}, \omega^{q}) \leq \delta \left[ r_{c}^{k} C^{q}g_{C^{q}}^{D} + \alpha r_{c}^{k} \omega^{q}g_{\omega^{q}}^{D} \right] \),

2. \( C^{q}g_{C^{q+1}}^{D} + \omega^{q}g_{\omega^{q+1}}^{D} \geq \sigma \left[ C^{q}g_{C^{q}}^{D} + \omega^{q}g_{\omega^{q}}^{D} \right] \).

in which \( 0 < \delta < \sigma < 1 \). Applying the Wolfe conditions and setting parameters \( \delta \) and \( \sigma \) ensures that the deviation function \( D \) and its gradients \( g_{C}^{D} \) and \( g_{\omega}^{D} \) decrease sufficiently at each step. Usually \( \delta \) is chosen to be very small and \( \sigma \) to be very close to 1. Throughout this study these parameters are kept constant \( \delta = 0.01, \sigma = 0.9 \) using the suggestions in [53, 54, 106].

Figure 3.11: Schematic representation of a case of homogenous convergence. Left: The candidate periodic solution \( \bar{x} \) changes its shape in a manner that reduces the sum of the norms of the distance vectors \( d_{i} \), i.e. the deviation function \( D \). Right: As the deviation function \( D \) tends to zero, the candidate solution \( \bar{x} \) converges to periodic orbits of the system.
In this regard, defining

$$\Gamma(v^T) = \begin{bmatrix}
v^T & 0_{1 \times 2N} & \cdots & 0_{1 \times 2N} \\
0_{1 \times 2N} & v^T & \cdots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
0_{1 \times 2N} & \cdots & 0_{1 \times 2N} & v^T
\end{bmatrix},$$

(3.35)

and

$$\alpha_m^T = \begin{bmatrix}
\cos \phi_m & 2^2 \cos(2\phi_m) & \cdots & (N)^2 \cos(N\phi_m) \\
\sin \phi_m & 2^2 \sin(2\phi_m) & \cdots & (N)^2 \sin(N\phi_m)
\end{bmatrix}
$$

$$\beta_m^T = \begin{bmatrix}
\cos \phi_m & \cdots & \cos(N\phi_m) \\
\sin \phi_m & \cdots & \sin(N\phi_m)
\end{bmatrix}
$$

$$\phi_m = \frac{m}{M}(2\pi) \in [0, 2\pi), \ m = 1, \ldots, M.
\phi_{m+M} = 2\pi + \phi_m
$$

(3.36)

one can readily show that

$$\bar{x}^{(k)}_m = \Gamma(\beta_m^T) \bar{C}^{(k)}
$$

$$\bar{\delta}^{(k)}_m = \omega^2 \Gamma(\alpha_m^T) \bar{C}^{(k)} + f(\bar{x}^{(k)}_m)
$$

$$\bar{J}_{\alpha m}^{\delta^{(k)}} = \omega^2 \Gamma(\alpha_m^T) + \bar{J}_{\alpha}^f(\bar{x}^{(k)}_m) \Gamma(\beta_m^T)
$$

$$\bar{J}_{\omega m}^{\delta^{(k)}} = 2\omega \Gamma(\alpha_m^T) \bar{C}^{(k)}
$$

(3.37)

where $\bar{J}_u^v$ defines the Jacobian of the vector $u$ with respect to the vector $v$.

Therefore, the gradient vectors can be found as

$$g_C = \nabla_C D = \nabla_C \sum_{i=1}^{2M} (\delta_i^T \delta_i) = 2 \sum_{i=1}^{2M} (\bar{J}_C^\delta)^T \bar{\delta}_i
$$

$$g_\omega := \nabla_\omega D = \nabla_\omega \sum_{i=1}^{2M} (\delta_i^T \delta_i) = 2 \sum_{i=1}^{2M} (\bar{J}_\omega^\delta)^T \bar{\delta}_i.
$$

(3.38)

### 3.5.2 Newtonian MMC

One can also use a Newtonian algorithm to minimize the vector $\delta$ by modifying the candidate solution as
\[ C^{q+1} = C^q + \Delta C^q \]
\[ \omega^{q+1} = \omega^q + \Delta \omega^q. \]  

(3.39)

where \( \Delta C^q \) and \( \Delta \omega^q \) are computed by solving a set of overdetermined linear equations defined by

\[
\begin{bmatrix}
\bar{J} \delta q_i C_q
\bar{J} \delta q_i \omega_q
\\
\Delta C^q_i
\Delta \omega^q_i
\end{bmatrix} = \bar{\delta}^q_i, \ i = 1, ..., 2M
\]

(3.40)

Note that the constraint condition is enforced to guarantee uniqueness of the solution host (defined by \( C^q \) and \( \omega^q \)) in each iteration. The system in (3.40) can be further simplified by averaging to obtain

\[
\frac{1}{2M} \sum_{i=1}^{2M} \frac{\sum_{i=1}^{2M} \bar{J} \delta q_i C_q}{\sum_{i=1}^{2M} \bar{J} \delta q_i \omega_q} \begin{bmatrix}
\Delta C^q_i
\Delta \omega^q_i
\end{bmatrix} = \frac{1}{2M} \sum_{i=1}^{2M} \bar{\delta}^q_i
\]

(3.41)

which can be solved for \( \Delta C^q \) and \( \Delta \omega^q \) using the Moore-Penrose inverse. In another (more traditional) approach one can enforce uniqueness of the solution host by recasting the system in (3.40) as

\[
\begin{bmatrix}
\bar{J} \delta q_i C_q & \bar{J} \delta q_i \omega_q \\
\vdots & \vdots \\
\bar{J} \delta q_{2M} C_q & \bar{J} \delta q_{2M} \omega_q
\end{bmatrix} \begin{bmatrix}
\Delta C^q
\Delta \omega^q
\end{bmatrix} = \begin{bmatrix}
\bar{\delta}^q_1 \\
\vdots \\
\bar{\delta}^q_{2M}
\end{bmatrix}
\]

which, due to good results obtained from solving the smaller problem in (3.41), was not pursued in this study.

### 3.6 Case Study I: A 2DOF System

This section first demonstrates how the MMC algorithm modifies a periodic solution host to match a periodic solution of a system. Then, we will examine convergence properties of the two proposed versions of the MMC algorithm and provide a comparison with the widely used Shooting algorithm in
terms of radius of convergence and speed. In this regard we choose a familiar two DOF system
\[
\begin{bmatrix}
\ddot{x}_1 \\
\ddot{x}_2
\end{bmatrix} =
\begin{bmatrix}
-2 & 1 \\
1 & -2
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} +
\begin{bmatrix}
-\frac{1}{2} & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
x_3 \\
x_4
\end{bmatrix},
\]
(3.42)
which has been extensively studied in \textcolor{red}{[117, 122, 81, 5]}. This oscillator is known to have branches of periodic orbits, sometimes called nonlinear modes of vibration that have been calculated both analytically and numerically\textcolor{red}{[5]}.

### 3.6.1 Homogenous Convergence

Figures \textcolor{red}{3.12} and \textcolor{red}{3.13} illustrate how the proposed MMC algorithm allows one to find the Coefficients of a Fourier series expansion of a periodic solution of the system and its period by gradually modifying both the Coefficients set and the period of the solution host. We refer to this type of convergence, where all the pairs of collocation points converge to one periodic solution of the system, as a homogenous convergence. In this example, the algorithm was initiated at the point \( y = [1.905, 0.9253, 0, 0]^T \) due to randomly generated coefficients of the initial Fourier series expansion, \( A_{ij}^{(0)} \) with \( B_{ij}^{(0)} = 0 \) and the control parameters set to \( r_c^{(0)} = 1e-6, r_w^{(0)} = 1e-8, N = 10, M = 20, \delta = 0.05 \) and \( \sigma = 0.9 \).

![Figure 3.12: The acceleration and force vectors, i.e. \( \ddot{x} \) and \( f(x) \), of the candidate solution.](image-url)
As shown in Fig. 3.12, the analytical acceleration of the solution host, i.e. \( \ddot{x} \), and its value provided by the vector field, i.e. \( f(x) \) generate disparate shapes initially. However after \( K = 36 \) steps in the direction provided by (3.30), those two converge to an identical shape. This convergence, as evident in Fig. 3.13, is a result of considerable change in both coefficients of the Fourier series (shape and amplitude) and the period (length) of the solution host. The initial guess, which had a period of \( T = 19.35 \), changes as the candidate solution navigates to a periodic orbit of the system, i.e. toward satisfying (3.27), until it converges to a periodic solution with the period \( T = 16.88 \).

![Figure 3.13: The displacement is shown for the candidate solution \( x_1^{(k)}(t) \) where it matches the true periodic orbit of the system with \( x_1^{(36)}(t) \).](image)

### 3.6.2 Radius and Speed of Convergence

The MMC algorithm seems to have a large radius of convergence especially using the conjugate gradient version. For instance, Fig. (3.14) shows that the candidate solution (in the previous example) converges to a periodic orbit that is quite far from its initial start point in the state space and its initial period.
Figure 3.14: The path of candidate solution in the state space as it converges to a periodic orbit of the system.

To further clarify this feature, we compare the radius and speed of convergence of the conjugate gradient version of MMC (MMC-CG), its Newtonian counterpart (MMC-N) and a shooting algorithm proposed in [105] for the same initial guess of initial conditions and periods. Tables 3.1 and 3.2 show the initial conditions, period and the results for two types of typical scenarios where the initial guesses are far from any or extremely close to a periodic solution of the system (3.42), respectively. In both comparison sets, all three algorithms, i.e. MMC-CG, MMC-N and Shooting, were initiated at the same initial guess.

**Set I**- In this set, the initial guess contains the initial conditions of a periodic solution with its (initial) period chosen far from the actual period \( T'(0) = 19.1337 \ll T^* = 40.85 \). In result, the MMC-CG was able to converge to different periodic solutions (including the initial guess with the correct period) depending on what values the user chooses for the algorithm’s parameters. For instance, by choosing a very small step size for the coefficients and a much larger step size for the period of the solution host, i.e. setting \( r_c = 1e - 12 \) and \( r_w = 1e - 3 \), the MMC-CG was able to find the correct period for the initial state. In contrast, as shown in Fig.3.14, choosing larger coefficient and smaller period step sizes, i.e., \((r_c, r_w) = \{(1.33e - 8, 1e - 6), (2e - 8, 1e - 8), (1.33e - 7, 1e - 8), (1e - 6, 1e - 8)\}\) and \((\delta, \sigma) = (0.05, 0.9)\), led to finding periodic solutions farther from the initial guess.
Figure 3.15: The path of candidate solution host in the state space as it converges to a periodic orbit of the system using the MMC-CG. For different sets of parameters, the MMC-CG algorithms converges to different solutions on different branches of periodic solutions previously found in [5, 7].

These extreme cases show how large the radius of convergence is for the MMC-CG, and while this proves the versatility of the algorithm, they come at the price of large number of steps (iterations) and slower CPU time as outlined in Table 3.1.
Table 3.1: Comparison Set I- All algorithms were initiated at $y^{(0)} = [3.8097, 2.7759, 0, 0]$, $T^{(0)} = 19.1337$ as the initial (equivalent) guess of initial conditions /period. A true periodic solution exists with the initial conditions $y^{(0)} = [3.8097, 2.7759, 0, 0]$ and the period $T^* = 40.85$. The MMC-CG converges to five different solutions provided different sets of step sizes. * The algorithm was terminated after 1200 iterations.

In the cases of the MMC-N or Shooting algorithms, if the algorithm converged, it always converged to the same solution regardless of its settings. However, because of the length of the solution host in the MMC algorithm, i.e. $2T^*$, the MMC-N (and also MMC-CG) are capable of (prone to) converging to symmetric periodic solutions, i.e. $x(t) = x(T^* - t)$, with half periods $T = T^*/2$. Case 2 in the Table 3.1, shows such a convergence. In this comparison set, the Shooting algorithm did not converge to any solution.

Set II- While the previous set was mainly concerned with the radius of convergence of the three algorithms, in this set, we focus on speed of convergence. All algorithms were initiated extremely close to a periodic solution of the system in (3.42), i.e. by applying a 0.005% relative error to the initial condition and the period of the solution. Since the MMC algorithms do not require integration over any period of time, both the CG and (specially) the Newtonian version, as evident in Table 3.2, prove to be faster than the Shooting algorithm although MMC-CG usually requires more steps to converge to a solution.
Table 3.2: Comparison Set II- All algorithms were initiated at \( y^{(0)} = [-7.891622796236744, -32.546617291309410, 0, 0] \), \( T^{(0)} = 4.4671749194952996 \) as the initial (equivalent) guess of initial conditions.

Another interesting point is that unlike MMC algorithms, the Shooting algorithms doesn’t always converge to a solution smoothly. Modifying many pairs of collocation points on a solution host close to a periodic solution in the MMC algorithm, as opposed to only one in the Shooting algorithm, provides a much smoother convergence to the that solution. Figure 3.16 shows that, unlike the Shooting algorithm, both MMC algorithms (initiated at the same initial conditions/period) converge to the closest solution. Moreover, based on the results not shown here, it seems that for periodic solutions with larger periods, the CPUTime of the Shooting algorithm increases super-linearly while the MMC algorithms show sub-linear increases.

![Figure 3.16](image)

Figure 3.16: Unlike the Shooting algorithm, both MMC algorithms converge to the closest solution using 20 pairs of collocation points and \((r_c, r_w, \delta, \sigma) = (1e - 6, 1e - 8, 0.05, 0.9)\) parameters in MMC-CG.
3.7 Case Study II: A 10DOF System

To explore the scalability of the algorithm, it was also applied to a more complicated system with 10 DOF and thousands of parameters. This system also exhibits numerous cases of heterogeneous convergence and thus is an informative example. This system, shown in Fig. 3.17, consists of a geometrically nonlinear beam, that is modeled in the Abaqus software resulting in 123 degrees-of-freedom finite element model for the structure., coupled with a torsional spring at one end. Then, using an approach explained in [1], the dimension of the system is reduced. The mentioned procedure led to a ten-dimensional reduced order model that is described by

\[ \ddot{x} = f(x) = K_l x + \frac{1}{2} K_{nl1}(x_i)x + \frac{1}{3} K_{nl2}(x_i x_j)x, \quad i, j = 1, ..., 10, \]  

(3.43)

where \( K_l \) is a constant matrix defining the linear part of the vector field and \( K_{nl1} \) and \( K_{nl2} \) are function matrices that define the quadratic and cubic parts of the vector field, respectively.

Figure 3.17: The ten DOF system is a reduced order model of a geometrically nonlinear beam coupled with a linear torsional spring.
Figure 3.18: A periodic solution of the system (3.43) found after a homogeneous convergence with \( N = 10 \) and \( M = 50 \).

The proposed method was applied to this system and several cases of homogenous convergence were found using MMC-N with randomly generated set of initial Fourier series coefficients \( A^{(0)}_{ij} \) with \( B^{(0)}_{ij} = 0 \), such as the solution shown in Fig. 3.18.

### 3.7.1 Heterogeneous Convergences

However, the more interesting case is when the \( M \) collocation points on the solution host land on at least two different periodic orbits of the system. In such cases, although all the collocation points are guaranteed to be on periodic solutions of the system, because they satisfy (3.27), the candidate solution (and its period) will not match any of the final periodic solutions (or their periods). That is because, the periodic solution host (by construction) only satisfies the equation of motion at (all of) the collocation points, however, it is not guaranteed to satisfy it away from the collocation points. To explain such a situation, we present a case where the candidate solution converges to three different periodic orbits.

In this regard, Fig. (3.19) shows the \( x_1(t) \) component of the (final) candidate solution and five (three distinct) resultant periodic orbits. This shows that, although all 5 points satisfy the condition (3.27), they converged on three different periodic orbits. Furthermore, as stated before, the (final) candidate solution does not (and cannot) match either of the periodic solutions.
More interestingly, Table 3.3 shows that one finds even more periodic solutions as the number of pairs of collocation points increases. Furthermore, it seems that increasing the number of collocation points affects the CPUTime and the number of iterations in a sub-linear manner, if at all. Figure 3.20 shows the periodic solutions corresponding to the first pairs of collocation points in the second and third cases represented in Table 3.3 respectively.

<table>
<thead>
<tr>
<th>Case</th>
<th># Collocation Points</th>
<th># Sol’ns</th>
<th>CPUTime</th>
<th>Iter’ns</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
<td>3</td>
<td>0.0713</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>20</td>
<td>14</td>
<td>0.1406</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>50</td>
<td>37</td>
<td>0.2656</td>
<td>3</td>
</tr>
</tbody>
</table>

Table 3.3: Increasing number of collocation points leads to more periodic solutions on the solution host without usually affecting number of iteration required by MMC-N to converge. The number of harmonics was kept constant for all three cases at $N = 10$. 

Figure 3.19: A case of heterogeneous convergence. The 5 pairs of collocation points converge on three different periodic solutions.
Figure 3.20: Phase portraits of the first periodic solutions on the solution hosts obtained in cases 2 and 3 in Table 3.3 with respectively 14 and 37 pairs of collocation points. The solutions shown are obtained by using the first collocation points, of 14 and 37 respectively, as initial conditions for the reduced order model. Left: case 2. Right: Case 3

It is interesting to consider what regions of the solution space are captured in each set of solutions of the MMC algorithm. One convenient way of characterizing the solution space is to use a frequency-energy plot where the frequency and total conserved energy in the solution are shown. Continuation algorithms are often used to determine how the frequency and energy of the low-energy linear modes of the system evolve with increasing energy, and all of this information can be readily shown on the frequency energy plot.

All periodic solutions of any system that (locally) satisfies the Implicit Function Theorem (IFT) will generate a continuous branch of periodic solutions [79, 30, 67, 125]. Continuation is a numerical process where a slightly changed state of a known solution is used to approximate the next solution on a continuous branch of periodic solutions[2, 34, 42, 86, 105]. The set of known solutions are very limited and usually only includes the periodic orbits of the underlying linear linear system at very low energies. In this sense, the MMC algorithm can be extremely helpful, first, by providing a rich set of initial known solutions that is not limited to the periodic orbits of the underlying linear system. Second, since it does not require integration over any period of time, it can also improve the speed of such a
process by replacing the Shooting algorithm that has traditionally been the underlying search method at each step\cite{34, 41, 81}.

Figure 3.21: Frequency-energy representation of periodic solutions of the system in Eq. (3.43) using a Shooting-based continuation algorithm (6 branches of periodic solutions) and MMC-N. Multiple discrete solutions, found on each solution host, are shown with similar markers.

In this regard, Fig. 3.21 shows 6 continuous branches of periodic solutions obtained using a Shooting-based continuation algorithm \cite{105}. Then MMC-N was used to obtained multiple discrete solutions on each branch. In order to initiate MMC-N directly on a branch, the initial conditions of a linear periodic solution $\bar{y}_0$ from that branch were used to generate a constraint on the initial defining coefficients of the solutions host, i.e. $\sum_{j=1}^{N} A_{ij} = \bar{x}_{i(0)}$, $\sum_{j=1}^{N} B_{ij} = \frac{T^*}{2\pi} \bar{x}_{i(0)} i = 1, ..., n$ where $T^*$ is the period of the mentioned solution. The coefficients then were perturbed as explained for simulations performed in Set II of Section 3.6.2 (with 0.01 relative error). This way, the MMC-N algorithm was initiated on the aforementioned branches of periodic solutions and close to the linear solution with the highest energy with 45 pairs of collocation points. Note that, by initiating the MMC algorithm around a solution with the highest energy in the (locally) linear regions\footnote{Any region with a very small change in fundamental frequencies of periodic solutions. At lower energies, these regions are quite large and as the energy of the solutions increase, they become smaller.}, one allows the collocation points to converge on
solutions with lower energies that share the same fundamental frequencies. This procedure was repeated 8 times resulting in 8 solution hosts, where each contained between 4 to 17 distinct periodic solutions. The solutions found on each of the mentioned solution hosts are shown with alike markers on top of their corresponding branches of periodic solutions in Fig. 3.21. Moreover, 7 sets of solutions obtained using MMC-N with random initial settings reveal cases of internal resonances (shown in the magnified view). These solutions are easily overlooked by shooting-based continuation algorithms; continuation tends to follow only the primary branches [5].

For instance, region A in Fig. 3.21 contains two periodic solutions found with MMC-N. The first solution, marked with a circle in Fig. 3.21 is shown in Fig. 3.22. This solution is identical to one found by the shooting-based continuation and represents a periodic solution associated to the first mode of vibration of the system (3.43). The second periodic solution, marked with a star in Fig. 3.21 is shown in Fig. 3.23 and exhibits a series of internal resonances ranging from 1:5 internal resonance (between the first and the fifth coordinate) to 1:18 internal resonance between the first and the eighth coordinates as evident in the provided time response. However, this solution cannot be found using a Shooting-based continuation algorithm unless the bifurcation point and the type of bifurcation are accurately known[5, 7]. Furthermore, since the MMC algorithm is capable of finding multiple solutions using one shared solution host it is intrinsically compatible with discovering multiple branches of periodic solutions at once. Moreover, as shown in the previous sections, since it can find multiple solutions in a fraction of time required by Shooting algorithm (for only one solution) and because of its sub-linear scalability, it is expected to considerably improve the speed of any continuation process.
Figure 3.22: Phase portraits of the periodic solution \( T = 0.02225 \text{ sec} \) of the system in Eq. (3.43) marked with a circle on Branch 1 in region A.

Figure 3.23: Left: Phase portraits of the second periodic solution \( T = 0.023 \text{ sec} \) of the system in Eq. (3.43) in region A. Right: This solution represents a case with multiple internal resonances. The time history of the response reveals cases of 1:5 and 1:18 internal resonances between the first and the fifth and also the first and the eighth coordinates respectively.
Chapter 4

Stability of Nonlinear Modes

In this section, accurate stability analysis and of nonlinear modes and its validation is explained. In order to determine the stability of nonlinear modes (periodic orbits) of nonlinear systems, one needs to perform Floquet analysis. In this regard, Floquet theory is explained briefly next.

The behavior of a nonlinear dynamical system can often be explained by studying its fixed points and periodic orbits [72, 62, 78]. In the case of oscillatory dynamical systems, the role of the latter form of equilibrium becomes more prominent. In this regard, many have proposed methods to find (branches) periodic solutions of nonlinear systems [40, 29, 37, 101, 128, 105, 5] which also led to computational software packages such as AUTO [34], MatCont [41], CONTENT [90], LINLBF [82] and BIFPACK [120]. In order to find new (branches of) periodic solutions, one must be able to detect different types of bifurcations of periodic solutions which may generate new (branches of periodic solutions. All of the packages mentioned above determine the stability of periodic solutions as the solutions are found. The methods used in these packages are well established and can be performed to accurately detect most bifurcations.

For example, multiple methods of stability and bifurcation analysis of periodic solutions developed by Doedle et al [42, 25, 45, 43, 44, 46, 47] are incorporated in AUTO. More methods have been proposed
by Guckenheimer and Cho [64, 35]. Even the more complex case of finding global bifurcations of (homoclinic/heteroclinic) periodic solutions where their periods tend to infinity was extensively studied which led to two classes of methods by Doedel and Kernevez [43, 44] and Beyn [22, 23] wherein one searches for either periodic solutions with very large but fixed periods or truncates the infinite period to a finite interval and then imposes an asymptotic boundary-value condition. The approximation error caused by the mentioned truncation in the second class of such methods, has been also studied by Hoog and Weiss [39], Markowich [95] and Beyn [24].

However, there are problems for which these algorithms prove inadequate. For example, it is difficult to computationally predict when certain bifurcations, such as Hopf-Hopf and Andronov-Hopf [96, 15, 89] bifurcations occur, and these frequently occur in oscillatory systems. These bifurcations are difficult to predict because the Floquet exponents\(^1\) associated with the periodic solutions are very close to zero, and hence they are sensitive to numerical error. The most prominent sources of such can be categorized as error accumulated by numerical integration, round-off error introduced by solving eigenvalue problems or the approximation error in identifying the periodic solutions [56, 51, 47, 22, 46]. One approach is to perform the stability analysis with the maximum precision allowed by the machine, but that is inefficient and may still may prove inadequate to accurately predict all of the bifurcations. In practice most practitioners usually perform stability/bifurcation analysis up to a lower (and usually fixed) level of accuracy.

This study provides error bounds for integration error as well as any error in approximating the periodic solution. Then these bounds are used to augment an existing algorithm that finds periodic solutions by performing stability analysis at each point to check whether the resultant Floquet exponents fall outside the error bounds. If they do, then this provides assurance that an accurate stability analysis, and therefore a reliable bifurcation analysis, is possible. One can also use these error bounds to adapt

\(^1\)For periodic solutions, Floquet multipliers are eigenvalues of Poincare maps and Lyapunov exponents are the same as the real parts of the Floquet exponents.
the error tolerance on the algorithm used to calculate the periodic solutions or the Floquet exponents to avoid wasting computational effort. This can dramatically reduce the computational cost when compared to an approach where a uniform (and typically very small) error tolerance is used. Alternatively, if the periodic solutions have already been found then this process can be used to determine which Floquet exponents can be trusted. Note that the proposed validation process does not replace or provide an alternative for finding periodic solutions or performing stability analysis, but simply adds an extra step to an existing algorithm.

Floquet theory provides a sound theoretical basis for finding the aforementioned error bounds and therefore is outlined in the next section. Then, upper limits are derived for errors caused by accumulation of integration error and approximation of the periodic solutions. Section 3 demonstrates the application of the mentioned error bounds on a simple example. Section 4 discusses the advantages and also limitations of the proposed algorithm. Conclusions and some remarks are given in section 5.

4.0.2 Floquet Theory at a Glance

The stability of periodic solutions of a nonlinear system is typically assessed by analyzing the solutions of the linearization of the system around its periodic orbit [14, 141]. In this sense, consider a system of autonomous ordinary differential equations having the following form:

\[
\ddot{x} = f(x), \quad x \in \mathbb{R}^n
\]  

(4.1)

where \( f : U \to \mathbb{R}^n \) is \( C^r \), \( r \geq 1 \) with \( U \) an open set in \( \mathbb{R}^n \) and \( \dot{x} = d^2x/dt^2 \). We can rewrite this system in the space of dependent variables \( x \) and \( y = dx/dt \), often referred to as phase space, as

\[\text{This study uses a system of the form } \ddot{x} = f(x) \text{ as an example to illustrate the results of the proposed expeditious validation algorithm. However, as it is shown in Appendix 4, the algorithm can be readily adjusted so it can be used in Floquet analysis of systems defined by } \ddot{x} = f(x, dx/dt) + g(t) \text{ where } g : U \to \mathbb{R}^n \text{ is also } C^r, \ r \geq 1 \text{ with } U \text{ an open set in } \mathbb{R}.\]
\[
\dot{z} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} y \\ f(x) \end{bmatrix} = F(z).
\] (4.2)

Furthermore, suppose this system accepts a periodic solution represented by \(\bar{z}(t)\). Then, using the coordinate transformation \(z = \tilde{z} + \bar{z}\), and after Taylor expansion of \(F(\tilde{z} + \bar{z})\) about \(\tilde{z} = 0\), one can replace (4.2) with its linear time varying approximation about the periodic solution as

\[
\dot{\tilde{z}} = \mathcal{J}F(\tilde{z})\tilde{z}.
\] (4.3)

Since \(\bar{z}\) has a period \(T\), so does the Jacobian matrix \(\mathcal{J}F(\tilde{z})\). Then, according to Floquet theory, if all entries of the matrix \(\mathcal{J}F(\tilde{z})\) are (Lebesgue) integrable on \([0,T]\), then a \(2n \times 2n\) matrix \(Z(t)\) with continuous entries exists that satisfies

\[
\frac{dZ}{dt} = \mathcal{J}F(\tilde{z})Z, \quad Z(0) = I_{2n}
\] (4.4)

where \(I_{2n}\) is the identity matrix of order \(2n\) [55, 49]. The matrix \(Z(t)\) is called the transition matrix and it satisfies

\[
Z(t + T) = Z(t)Z(T), \quad t \in \mathbb{R},
\] (4.5)

wherein the matrix \(Z(T)\) is known as the Monodromy matrix and its eigenvalues \(\rho_i\) are called the Floquet multipliers of the periodic solution \(\bar{z}\) (of the nonlinear system in Eq. (4.2)). Also, according to the Floquet theory, the transition matrix accepts the form

\[
Z(t) = W(t)e^{tM}, \quad W(0) = I_{2n}
\] (4.6)

where, \(W(t)\) is a continuous periodic matrix with period \(T\) and \(M\) is a constant matrix, both of the order \(2n\). Consequently, one can also write
The eigenvalues $\mu_i$ of the matrix $M$ are called the Floquet exponents and using the above equation can also be represented as

$$e^{T \mu_i} = \rho_i, \ i = 1, ..., 2n. \quad (4.8)$$

Hence, Floquet theory asserts that the system in Eq. (4.3) has a set of generalized eigensolutions of the form

$$\tilde{z}_{i,k} = e^{\mu_i t} Q_{i,k}(t)$$

$$Q_{i,k}(t) = \sum_{j=1}^{m_i} \left[ \frac{t^{j-1}}{(j-1)!} q_{i,j}(t) \right], \ k = 1, ..., m_i$$

$$\sum_{i=1}^{N \leq 2n} m_i = 2n \quad (4.9)$$

where $q_{i,j}(t)$ are continuous, $T$ periodic vector functions and $m_i$ represent the algebraic multiplicity of the Floquet exponents $\mu_i$. For example, if $\mu_i$ has an algebraic multiplicity of one, i.e. is simple, then $m_i = 1$ and $\tilde{z}_{i,q_1}(t) = e^{\mu_i t} q_i(t)$. However, if $\mu_i$ is non-simple with a multiplicity of $m_i = 3$, i.e. it forms a $3 \times 3$-cell in the Jordan form of $M$, then the system in Eq. (4.3) accepts a set of three eigensolutions

$$\tilde{z}_{i,1}(t) = e^{\mu_i t} q_{i,1}(t),$$

$$\tilde{z}_{i,2}(t) = e^{\mu_i t} \left[ q_{i,1}(t) + t q_{i,2}(t) \right],$$

$$\tilde{z}_{i,3}(t) = e^{\mu_i t} \left[ q_{i,1}(t) + t q_{i,2}(t) + \frac{t^2}{2} q_{i,3}(t) \right]. \quad (4.10)$$

All periodic solutions always have at least one multiplier equal to 1 \(^{3}\). If all of other multipliers reside inside the unit circle in the complex plane, then the periodic orbit is asymptotically stable. If at least one multiplier sits outside the unit circle, then the periodic orbit is unstable. In all other cases, where more than one multipliers are situated on the unit circle, further investigation is required. Specially, when calculating the Monodromy matrix along a branch of periodic orbits, three cases of singularities can

\(^{3}\)Its eigenvector is tangent to the periodic orbit $\tilde{z}(t)$ at the point $t = 0$. 

$$Z(T) = e^{TM}. \quad (4.7)$$
Figure 4.1: The perturbed periodic solution \( \tilde{z} = z - \bar{z} \) will asymptotically converge to (will not diverge from) the periodic solution \( \bar{z} \) if the solution of linear system (4.3) is stable (marginally stable), i.e. the response to the perturbation term is bounded.

occur, namely (1) a fold singularity, when the multiplier 1 has algebraic multiplicity \( m \geq 2 \), (2) a flip singularity, when there is multiplier equal to \(-1\) and (3) a Neimark-Sacker singularity, when there is a conjugate pair of complex multipliers on the unit circle\([62, 89]\). The stability can be equally determined using the Floquet exponents associated with the periodic orbit. If the Floquet exponents of the system in Eq. (4.3) are all simple, as depicted in Fig. 4.1, they can be used to study the boundedness of its solutions. This provides insight into the stability of the periodic orbit of the original nonlinear system in Eq. (4.2).

4.1 Error Bounds in Numerical Stability Analysis

In one approach to calculate the Floquet exponents of the system (4.3), one can calculate \( 2n \) linearly independent solutions of the system (4.3) and then, construct the matrix \( \tilde{Z}(t) = [\tilde{z}_1 : \cdots : \tilde{z}_n] \) which by definition satisfies Eq.(4.4). Since the system (4.3) is linear, linear independence of the \( 2n \) solutions can be guaranteed if one uses the columns of the identity matrix as initial conditions. This way, one can find the Monodromy matrix as \( \tilde{Z}(T) \) and it’s eigenvalues as Floquet multipliers \( \rho_i \) (and consequently \( \mu_i \)) and it’s the generalized eigenvectors \( \tilde{z}_i \).
The results of Floquet analysis is prone to approximation error that may stem from various sources: approximation error when finding a periodic solution, accumulation of integration error or error introduced by solving an eigenvalue problem. It is worth mentioning that not all algorithms use direct integration to evaluate stability. For example, some use a piecewise constant, linear or quadratic approximation of $J_{F}(\bar{z})$ \[126, 127\] or use a polynomial approximation (and updating) of the characteristic equation of the Monodromy matrix \[73, 127, 85, 139\]. These approaches may be preferable in some applications, but other works have shown that these approaches are not suitable for accurate bifurcation analysis which involves near-zero Floquet exponents \[111, 121\]. In this section, we quantify the effects of the first two sources of numerical approximation errors. The effect of round-off error has been studied before \[51\] and is significant when the Monodromy matrix has conditioning problem and it has been addressed by deflation of large (infinity) eigenvalues \[119\].

**Initial error**- Since for most systems an exact, analytical solution for the periodic orbits is not known, we consider both the periodic solution $\bar{z}$ and its period $T$ to be prone to approximation errors. This approximation will affect the Floquet analysis when calculating $J_{F}(\bar{z})$ in Eq. (4.3) and also the solutions of the linearized system.

**Integration error**- Integration error depends on the accuracy of integrator that is used. The accuracy of an integrator is defined by the maximum error allowed in a converged solution. In other words, the error $E$ in the solution, carried out by (most common) integrators (including all of the Runge-Kutta integrators in MATLAB such as ode45), must satisfy $E \leq h$ where the vector $h = \max\{\text{RelTol} \times |x|, \text{AbsTol}\}$ is the integrator tolerance. The vectors $\text{RelTol}$ and $\text{AbsTol}$, respectively, specify the maximum relative and absolute error tolerances accepted for all component of the solution. Hence, if $|x(i)| \approx 1$, then $h$ determines the number of correct digits in all solution components except those smaller than the thresholds $\text{AbsTol}$. For example, setting $h = \max\{1e-6[1,...,1]^T, 1e-12[1,...,1]^T\}$ means that at least 6 correct digits are requested in each component of the solution while all components smaller than
$1e-12$ are essentially considered as zeros.

In this study, however, we are interested in a tolerance settings that allows one to assure that the maximum integration error in the solution does not depend on the form of the solution. In order to grant such an assumption, we must set the tolerance in a way that $h = \max\{\text{RelTol} \cdot |x|, \text{AbsTol}\} = \text{AbsTol}$. In this regard, as shown in Appendix 5, setting $\text{RelTol}$ to be equal to $\text{AbsTol}$, will ensure that $h$ will always be equal to $\text{AbsTol}$, i.e. $h = \text{AbsTol}$.

### 4.1.1 Initial and Integration Errors Bounds

The general form provided in Eq. 4.9 implies that for each distinct Floquet exponent $\mu_i$ there exists at least one generalized eigensolution of the linear system in Eq. 4.3 of the form

$$\tilde{z}_i(t) = e^{\mu_i t} q_i(t), \ i \in \mathbb{N}, \ i \leq 2n.$$  

Note that, in the case of algebraic multiplicity of $m_i > 1$, the mentioned eigensolution will be $\tilde{z}_{i,1}$. Consequently, one can assume that each distinct Floquet exponent $\mu_i$ of the linear system in Eq. 4.3, when it is rearranged in its second-order-form as

$$\ddot{\tilde{u}} = \mathcal{J} \bar{x}(\bar{x}) \tilde{u},$$  

(4.11)

is associated with a generalized eigensolution of the form

$$\tilde{u}_i(t) = e^{\mu_i t} q_i(t), \ i \in \mathbb{N}, \ i \leq 2n,$$  

(4.12)

where $\tilde{z}_i = \begin{bmatrix} \tilde{u}_i \\ \tilde{v}_i \end{bmatrix}$. This way, defining the perturbation operator $\delta(.)$ as

$$\delta(.) = \sum_{k \in \mathbb{N}} \frac{1}{k!} \left[ \sum_{m=0}^{k} \binom{k}{m} \frac{\partial^n (.)}{\partial T^m} \frac{\partial^m (.)}{\partial \mu^{k-m}} (\delta T)^m (\delta \mu)^{k-m} \right]$$
one can write

\[
\delta \tilde{u}_i = \frac{\partial \tilde{u}_i}{\partial \mu_i} \delta \mu_i + \frac{\partial \tilde{u}_i}{\partial T} \delta T + (HOT) \simeq t \tilde{u}_i(t) \delta \mu_i + \frac{\partial \tilde{u}_i}{\partial T} \delta T
\]  

(4.13)

which reveals that, even with only the linear perturbation terms, the discrepancy in the Floquet exponent, i.e. \(\delta \mu_i\), can not be identified using only the calculated solution \(\tilde{u}_i(t)\), therefore, further assumptions and analysis are required.

Since the system (4.3) is linear, the approximation error in its generalized eigensolution can be considered as the initial error plus the error accumulated by integration over one period. In order to benefit from this property, define \(u^{(1)}_i = e^{\mu_i^{(1)} t} q^{(1)}_i(t)\) and \(u^{(2)}_i = e^{\mu_i^{(2)} t} q^{(2)}_i(t)\) as the \(i\)-th generalized eigensolutions of the system (4.3), respectively, with no integration error and no initial error. In other words

- \(\delta u_i^{(1)}|_{t=0} = \delta \tilde{u}(t)|_{t=0} = \alpha \in \mathbb{R}^n\)
- \(\delta u_i^{(2)}|_{t=0} = 0, \delta u_i^{(2)}|_{t=T} = \delta \tilde{u}(t)|_{t=T} - \delta u_i^{(1)}|_{t=T}\)

It’s easy to show that \(\delta u_i^{(1)}\) and \(\delta u_i^{(2)}\) are linearly independent. Moreover by definition

\[
\delta \tilde{u}_i|_{t=T} = \delta u_i^{(1)}|_{t=T} + \delta u_i^{(2)}|_{t=T}.
\]  

(4.14)

The above equation can be expanded using linear perturbation terms and also using

- \((t \tilde{u}_i \delta \mu_i)|_{t=T} = e^{\mu_i^{(1)} T} T \tilde{u}_i|_{t=0} \delta \mu_i^{(1)}\)
- \((t u_i^{(1)} \delta \mu_i)|_{t=T} = e^{\mu_i^{(1)} T} T u_i^{(1)}|_{t=0} \delta \mu_i^{(1)}\)
- \((t u_i^{(2)} \delta \mu_i)|_{t=T} = e^{\mu_i^{(2)} T} T u_i^{(2)}|_{t=0} \delta \mu_i^{(2)}\)
\[ e^{\mu_i T} \bar{u}_{|t=0} \delta \mu_i + \frac{\partial \bar{u}_i}{\partial T} \delta T \simeq \left( e^{\mu_i (1) T} u_{i|t=0}^{(1)}(1) + \frac{\partial u_i^{(1)}}{\partial T} \delta T \right) + \left( e^{\mu_i (2) T} u_{i|t=0}^{(2)}(2) + \frac{\partial u_i^{(2)}}{\partial T} \delta T \right). \]

Furthermore, since

\[
\left( \frac{\partial \bar{u}_i}{\partial T} \delta T \right)_{|t=T} = \left( e^{\mu_i (1) T} \frac{\partial q_i}{\partial T} \delta T \right)_{|t=T} = e^{\mu_i (1) T} \left( \frac{\partial q_i}{\partial T} \delta T \right)_{|t=0} = e^{\mu_i (1) T} \left( \frac{\partial q_i}{\partial T} \delta T \right)_{|t=0} = 0
\]

one can obtain

\[ e^{\mu_i T} \bar{u}_{|t=0} \delta \mu_i + e^{\mu_i T} \left( \frac{\partial \bar{u}_i}{\partial T} \delta T \right)_{|t=0} \simeq e^{\mu_i (1) T} u_{i|t=0}^{(1)} + e^{\mu_i (1) T} \left( \frac{\partial \bar{u}_i}{\partial T} \delta T \right)_{|t=0} + e^{\mu_i (2) T} u_{i|t=0}^{(2)} \delta \mu_i. \]

Moreover, since \( u^{(1)}, u^{(2)}, \mu^{(1)} \) and \( \mu^{(2)} \) can be also described as \( u_i^{(1)}|_{t=0} = \bar{u}|_{t=0}, u_i^{(2)}|_{t=0} = \tilde{\bar{u}}|_{t=0} - \delta \bar{u}(t)|_{t=0}, \mu^{(1)} = \mu_i + \delta \mu^{(1)}_i \) and \( \mu^{(2)} = \mu_i + \delta \mu^{(2)}_i \), we can further simplify the above equation by omitting the higher order perturbation as

\[ T \bar{u}_{|t=0} \delta \mu_i \simeq T \bar{u}_{|t=0} \delta \mu^{(1)}_i + T \tilde{\bar{u}}_{|t=0} \delta \mu^{(2)}_i. \]

and consequently

\[ \delta \mu_i \simeq \delta \mu^{(1)}_i + \delta \mu^{(2)}_i. \]  \hspace{1cm} (4.15)

Therefore, one can conclude that
\[ |\Re\{\delta \mu_i\}| \leq |\Re\{\delta \mu_i^{(1)}\}| + |\Re\{\delta \mu_i^{(2)}\}| \quad (4.16) \]

The following theorem provides the above approximate bounds for \( \delta \mu_i^{(1)} \) and \( \delta \mu_i^{(2)} \).

**Theorem 4.1-** Suppose that \( \tilde{z}_i = \begin{bmatrix} \tilde{u}_i \\ \tilde{v}_i \end{bmatrix} \) is the generalized eigenvector of Monodromy matrix associated with \( \mu_i \) of the system (4.3). Then, assuming that the Floquet exponent \( |\Re [\mu_i]| \ll 1 \), then

\[
|\Re\{\delta \mu_i\}| \leq \frac{1}{2} \left| \Re \left\{ \left[ \left( \frac{\partial f}{\partial x}(\bar{x}(T)) \right) \tilde{u}_i \right] \delta T + \left[ J_f(\bar{x}(T)) - \mu_i^2 I \right] \delta \tilde{u}_i \right\} \cdot \tilde{v}_i \right|^{\frac{1}{2}} + \left| \Re \left\{ \frac{T}{\|\tilde{v}_i\|^2} \left[ \| J_f(\bar{x}) \| \tilde{u}_i + 2 \tilde{v}_i \right] \cdot \left[ \frac{1}{\| J_f(\bar{x}) \|} \right] \right\} \right| \quad (4.17)
\]

where \( h = \max\{\text{RelTol} \cdot |x|, \text{AbsTol}\} \) is the integration tolerance.

**Proof.** See Appendix 3

### 4.2 Iterative Validation Algorithm

In this section, a stronger form of the inequality in Eq. (4.16), i.e.

\[ |\Re\{\delta \mu_i\}| \leq L_1 + L_2 \quad (4.18) \]

is used as the criteria to validate numerically computed Floquet exponents. The validation process is explained through a simple example using a periodic solutions of a two DOF system (shown in Fig.(4.2))

defined by

\[
\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 0.5x_1^3 \\ 0 \end{bmatrix} \quad (4.19)
\]

where \( k_1 = k_2 = k_3 = 1 \) and \( k_{nl} = 0.5 \).
In this regard, we utilize the mentioned criteria as the termination criteria in calculating Floquet exponents explained in the proposed validation process illustrated in Fig. 4.3. In other words, the main function of the criteria (4.16) is to determine when a more accurate approximation is required and in which step of the process of calculating Floquet exponents.

Starting from the top, the first two blocks represents the process of approximating the periodic solutions and their periods where the variables $f$, $J^f_x$, $T$ and therefore $\delta J^f_x = J^f_x(x_{k+1}) - J^f_x(x_k)$ and $\delta T = T_{k+1} - T_k$ are required. The indices $k$ and $k + 1$ represents two consecutive steps as the algorithm iterates towards a periodic solution. These variables are, or can be readily calculated from, natural products of all algorithms of finding periodic orbits. For example, all variations of the Shooting algorithm, used in AUTO, MATCONT and in [128, 105], calculate these variables at each step while searching for periodic orbits. Therefore, this algorithm can be added to periodic orbit solvers without imposing a significant computational burden. Moreover, since these variables are only needed when the periodicity of the solution $\bar{x}$ is satisfied, the validation process only requires continuing the the process for one (or few more in case of invalid exponents) steps without the need to start over. For example, consider the periodic solution defined by the initial conditions $\bar{z}_0^{(k)} = [-3.6381344815293.91069030000]^{T}$, the period $T^{(k)} = 4.455831478089112$ and the termination criteria $\epsilon = \frac{\|z(T,\bar{z}_0) - \bar{z}_0\|}{\|\bar{z}_0\|} \leq 1e - 10$. In order to validate the Floquet exponent associated to this periodic solution, the algorithm is required to also report the next approximation of $\bar{z}_0$ and $T_0$ as $\bar{z}_0^{(k+1)} = [-3.6381344818293.91069032000]^{T}$ and $T^{(k+1)} = 4.455831478088214$. Note that, finding
\( \bar{z}_0^{(k+1)} \) and \( T^{(k+1)} \), in Shooting-based algorithms (such as [105]) or other Solvers (see [5, 8]), does not require any change in \( \epsilon \). The required solutions can be found from the last and second to last iterations of the solver as described below.

The third block generates the response to the linearized system (4.3). This can be performed by solving the concatenated system

\[
\begin{align*}
\dot{z} &= F(z) \\
\dot{z} &= JF(\bar{z}(t))z
\end{align*}
\]

with the initial conditions defined by

\[
\begin{bmatrix}
\dot{z}(0) \\
\bar{z}(0)
\end{bmatrix}
= \begin{bmatrix}
\bar{z}_0^p \\
\bar{z}_0 \\
e_m
\end{bmatrix}
\quad m = 1, \ldots, 2n
\]

for the last two steps of the search for \( \bar{z}_0^p \) in the periodic solver, i.e. for \( \bar{z}_0^p, p = k, k+1 \). In this regard, the integrator will generate \( 2 \times (2n) \) linearly independent solution of the system (4.20) for \( \bar{z}_0^{(k)} \) and \( \bar{z}_0^{(k+1)} \) from zero to, respectively, \( T^{(k)} \) and \( T^{(k+1)} \). Note that, although it’s beyond the scope of this study, the efficiency of the above process, i.e. finding the fundamental matrix \( Z(t) \), can be dramatically improved by using a single-pass integration method [56, 57] (instead of \( 2n \)-pass in this study) or a predictor-corrector scheme [57, 31] without sacrificing the credibility of the results of Theorem 4.1.\(^4\)

Next, the two corresponding Monodromy matrices are formed and fed in to the eigenvalue problem solver which will solve two eigenvalue problems and ultimately generate \( [\mu_i]_k \), \( [\mu_i]_{k+1}, i = 1, \ldots, 2n \) and their corresponding generalized eigenvectors

\[
\begin{bmatrix}
\tilde{u}_i \\
\tilde{v}_i \\
\end{bmatrix}_k, \quad \begin{bmatrix}
\tilde{u}_i \\
\tilde{v}_i \\
\end{bmatrix}_{k+1}, \quad i = 1, \ldots, 2n
\]

The conditional block in Fig. 4.3 simply checks if the criteria (4.16) is satisfied. If not, the convergence settings for the periodic orbit solver and/or integrator will be accordingly adjusted. In other words, one must reduce \( \epsilon \) and/or \( h \) depending on the value of \( L_1 \) and \( L_2 \) respectively. For instance, since \( L_1 \) corresponds to periodic solution error and is governed by \( \epsilon \), then if \( L_2 \ll |\Re\{\mu_k\}| \leq L_1 \), one would only need to

\(^4\)Note that, if the periodic solution \( \bar{z}_p(t), t \in [0, T] \) was stored by the periodic solver, then the lower of the two equations in (4.20) could simply be used by interpolating on the solution \( \bar{z}_p(t) \).
reduce \( \epsilon \). Conversely, if \( L_1 \ll |\Re\{\mu_k\}| \leq L_2 \), then only the integration tolerance \( h \) needs to be reduced.

In the above example, using \( \epsilon = 1e - 10 \) for the periodic solver and also \( h = 1e - 11 \times [1 1 1 1]^T \) for the integrator leads to
\[
\mu_k = \begin{bmatrix} 0.00003243382 + 0.70505194575i & 0.00003243382 + 0.70505194575i \\ 0.09244250706 + 0.00000000000i & 0.09244250706 + 0.00000000000i \end{bmatrix}
\]
\[
\mu_{k+1} = \begin{bmatrix} 0.00003245191 + 0.70505194568i & 0.00003255161 + 0.70505194568i \\ 0.09244252766 + 0.00000000000i & 0.09244252766 + 0.00000000000i \end{bmatrix}
\]
\[
L_1 = 1.0e-07 \times [0.913409612011250 0.13633586282717 \\ 0.01009554934274 0.001577680528440]
\]
\[
L_2 = 1.0e-07 \times [0.264210924176928 0.26429592543431 \\ 0.282980706781220 0.727054840544578]
\]
\[
L_1 + L_2 = 1.0e-06 \times [0.117762053618818 0.117792951222615 \\ 0.029307625617549 0.072863252107302]
\]
\[
|\Re\{\mu_k\}| = \begin{bmatrix} 0.000032433822646 & 0.000032433822546 \\ 0.092442507062221 & 0.092442507062196 \end{bmatrix}
\]

and therefore, no adjustment to the tolerances of the periodic orbit solver or the integrator was needed. However, if necessary, the mentioned tolerances can be modified simply by

\[
\epsilon_{k+1} = \epsilon_k / 2, \quad h_{k+1} = h_k / 2.
\] (4.22)

Of course more elaborate schemes could be adopted to replace Eq. (4.22) to increase the efficiency of the algorithm. In practice this algorithm will be performed for many periodic solutions which have been obtained using continuation techniques. When this is the case, we can reduce the computational burden dramatically by predicting the tolerances needed for the next solution. When this is done, we have found that typically no adjustment is needed and the scheme becomes very efficient. This can be readily done for the algorithm proposed here. The same approach, used in the proof of Theorem 4.1 in Appendix 3, can be used to provide such prediction.

In this regard, the validated results of the Floquet analysis for the previous periodic solution, i.e. \( \tilde{u}_i, \tilde{v}_i, \mu_i^{(1)}, \mu_i^{(2)}, \delta\mu_i^{(1)} \) and \( \delta\mu_i^{(2)} \), can be used to approximate the required precision defined by \( \epsilon \) and \( h \) for the periodic solver and the integrator that will lead to valid Floquet analysis results. The equations (7.8) and (7.9) in Appendix 3, can respectively be used to obtain
\[ h_{\text{predicted}} = \min_i \left\{ T \bar{u}_i + 2 \left[ f_\lambda(\bar{x}) \right]^{-1} \bar{v}_i \delta \mu_i^{(2)} \right\} \] (4.23)

\[ \delta T_{\text{predicted}} = \min_i \left[ \kappa_i \cdot \left\{ 2 \bar{v}_i \delta \mu_i^{(1)} - \left[ f_\lambda(\bar{x}(T)) - \mu_i^2 I \right] \delta \bar{u}_i \right\} \right] \frac{\| \kappa_i \|^2}{\| \kappa_i \|^2} \]

\[ \kappa = \frac{\partial f_\lambda(\bar{x}(T))}{\partial T} \bar{u}_i, \quad i = 1, \ldots, 2n. \]

Note that, \( T, \bar{x}(0) \) and \( \bar{x}(T) \), in the above equations belong to the next periodic orbit. Equation (4.23) provides only the integration tolerance for the next Floquet analysis \( h_{\text{predicted}} \) directly. However, the tolerance on the periodicity of the next periodic solution \( \epsilon_{(p+1)} \) can be inferred as

\[ \epsilon_{\text{predicted}} = \epsilon_{\text{current}} \left( \frac{T}{\delta T} \right)_{\text{current}} \left( \frac{\delta T_{\text{predicted}}}{T_{\text{nextSolution}}} \right). \] (4.24)

When the tolerance approaches the machine precision it is generally not possible to improve the error tolerance further. In these cases, Equations (4.23) and (4.24) can be used to detect this scenario and terminate the algorithm to avoid additional computations that are likely to be fruitless. Specifically, when the predictions reveal that the Floquet analysis would require tolerances \( \epsilon_{\text{predicted}} \) and \( h_{\text{predicted}} \) smaller than what is attainable by the periodic solver or the integrator, those solutions are skipped and flagged as potentially inaccurate so that further investigation can be performed using some other approach if desired. This, as will be explained in Section 4.2, is especially important in bifurcation analysis.

For example, if Eq’s. (4.23) and (4.24) suggest that a valid Floquet analysis demands \( h_{\text{predicted}} \ll 2.254 \times 1e^{-14} \) which is smaller than the smallest tolerance that MATLAB’s ode functions can achieve, then Floquet analysis would not be meaningful. That is not only because the Floquet exponents will not be valid but also because they could skew the result of any consequent analysis such as bifurcation analysis. In such cases the algorithm that carries out the analysis cannot recognize the difference between zero and the reported values.
Table 4.1: Performance comparison between Floquet analysis with and without predictive validation.

<table>
<thead>
<tr>
<th></th>
<th><code># Periodic Orbits</code></th>
<th><code># Performed Floquet Analysis</code></th>
<th><code># Valid Floquet Analysis</code></th>
<th>CPU Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>w\out Pred. Alg</td>
<td>12461</td>
<td>12461</td>
<td>3358</td>
<td>58 hrs</td>
</tr>
<tr>
<td>w\ Pred. Alg</td>
<td>12461</td>
<td>3320</td>
<td>3320</td>
<td>28 m</td>
</tr>
</tbody>
</table>

Figure 4.4: Floquet exponents of the second branch of periodic orbits (obtained by continuation of the second linear mode). Both algorithms, i.e. with and without predictive validation, return an identical Floquet exponent plots. Only the region with non-zero Floquet exponents is shown.

The effect of employing the predictions provided by Eq’s. (4.23) and (4.24) on the performance of the proposed algorithm is examined for a branch of 12461 previously identified periodic solutions. Table 4.1 shows the drastic improvement after using the mentioned prediction to detect and avoid invalid Floquet analysis. By using this approach to predict the tolerances needed, not only we avoided fruitless Floquet analysis but we we also reduced the number of iterations in the periodic orbit solver and when performing Floquet analysis.

Moreover, note that there is a slight discrepancy between the numbers of valid Floquet analysis in the two cases. This indicates that not all predicted tolerances were accurate, however, it accounts for only 0.22 % of the total number of periodic orbits.
Figure 4.5: Cross-section of a Poincare map is shown for the periodic solutions of the system (2.5). The red and blue curves represent a branch or periodic orbits obtained using continuation of first and second linear modes respectively (see [5]). The rest of closed curves are products of global bifurcations from periodic orbits on the red and blue curves [7]. The frequencies of the closed branches of periodic solutions have been scaled to cross the the red and blue curves at bifurcation points.

4.3 Benefits and Limitations

In this section, we will use branches of periodic orbits from the oscillator defined in the previous section to point out some of the immediate benefits of the proposed algorithm as well as its limitations. This system is known to have branches of periodic orbits, sometimes called nonlinear modes of vibration. The system has been extensively studied and its periodic orbits have been calculated both numerically and analytically [5]. The periodic orbits of the system (2.5) are illustrated in Fig. 4.5 as a cross-section view of their Poincare map. In this regard, a Poincare section $\Omega$ is defined by $\Omega = \{(x_1, x_2, y_1, y_2) \in \mathbb{R}^4 | y_2 = 0\}$ where $\Omega$ defines a hyperplane in $\mathbb{R}^4$. Next, a cross-section of fixed points of the mentioned Poincare map defined by the plane $y_1 = 0$, a contour of the Poincare map, is shown in Fig. 4.5.
4.3.1 Accurate Stability Analysis

The most immediate benefit of the proposed algorithm is evident in accurate and verifiable stability analysis. Stability analysis is an important part of design, and also identifying the range of safe operation, of nonlinear systems. It can also be used to predict or explain which set of nonlinear modes can or cannot be realized experimentally, and this information can be valuable in planning tests to identify the nonlinear modes. The proposed validation algorithm directly addresses the reliability of the stability analysis. Without a validation process such as the one presented here, one has no way of knowing which Floquet exponents are accurate and which are not. This is especially important when the stability analysis returns somewhat unexpected results. For example, stability analysis of the first and the second branches of periodic orbits, shown in Fig. 4.6, reveals unstable periodic orbits in the region where one would expect the response to be linear (the fundamental frequency of the periodic orbit is almost the same as the linear natural frequency $\delta \omega \approx 0$) that contradicts the conventional assumption that in this region the system, like its underlying linear system, has to be marginally stable. Moreover, as with most linear systems, it is known that a (series of) double-hopf bifurcation at the origin (equilibrium point) leads to two (even number of) branches of periodic orbits [89, 16, 15, 65]. For this system, this phenomenon gives rise to the red and blue curves shown in Fig. 4.5.

However, for the first time, one can use the validation algorithm to capture the resultant unstable periodic solutions and confirm the mentioned double-hopf bifurcation without performing any analytical bifurcation analysis.

That is because the mentioned usable periodic solutions were detected by the paths of very small but valid Floquet exponents in the range of $1e-11 \leq |\Re\{\mu_i\}| \leq 2.7e-7$ (see Fig. 4.7) which is much smaller than tolerances ($\sim 1e-4$) that are commonly used. This can be extremely useful if the system
Figure 4.6: Stability analysis of the first and the second branches of periodic orbits. Top: Frequency-Energy plane and (a cross-section of) Poincare map representations of the first branch. Unstable periodic solutions are marked in red. Periodic solutions can be unstable even in the very low energy (linear) region. Bottom: Stability analysis along the second branch of periodic orbits.

Figure 4.7: Floquet analysis of the low energy periodic orbits implies two hopf bifurcations (a double-hopf bifurcation) at the equilibrium point at the origin. More periodic orbits at lower energies would be required to complete the Floquet analysis and confirm the double-hopf bifurcation, however, search algorithms for finding periodic orbits converge poorly at very low energies.
has a dense (with infinitely many) set of bifurcations in regions away from the origin (its fixed point) where it cannot be analytically studied.

### 4.3.2 Accurate Bifurcation Analysis

In a stability analysis, one is only interested in determining whether the system is stable or unstable about a periodic solution. In other words, marginally stable or very slightly damped (positive or negative) periodic solutions typically receive no attention. In contrast, marginally stable periodic solutions are at the center of any bifurcation analysis. That is because bifurcations always happen when there is a change in stability of periodic solutions as a defining parameter (such as their period) goes through a (small) continuous change. The continuous change in the stability state (from stable to unstable or from marginally stable to stable/unstable and vice versa) implies that all bifurcations must happen at periodic solutions (or fixed points) that, at least in one dimension, are marginally stable[89, 121, 125].

In other words, all bifurcation points are periodic solutions with at least one Floquet exponent with zero real part that are surrounded by other neighboring periodic solutions with very small (real part of) Floquet exponents. Floquet analysis of the neighboring periodic solutions is also very important as they determine the type of bifurcation at the marginally stable solution. Therefore, performing a valid Floquet analysis (especially for solutions with very small exponents) is essential to any credible bifurcation analysis. In addition, identifying and removing invalid exponents is as important as verifying the valid ones. That is because, as mentioned earlier, the path that each one of Floquet exponents of the surrounding periodic solutions takes is of great significance and any slight distortion caused by invalid Floquet exponents can change the results of bifurcation analysis dramatically.

---

5It is worth mentioning that, according to the principle of persistence of solutions [89], as the branch of periodic solutions continues to grow after a bifurcation, all the bifurcated periodic solutions will also start new branches of periodic solutions with periods that may show a sudden discontinuous change compared to the one of the branch they bifurcated from. However, in bifurcation analysis, one is only concerned with detecting the periodic solutions where bifurcations happen and also the type of bifurcations. The study of the behavior of a system around bifurcation points such as jump phenomena, like the mentioned sudden change in the period (or the dominant frequency) of a trajectory of the system, is the topic of catastrophe theory and beyond the scope of this study (see [15, 16]).
Invalid Floquet multipliers (or exponents) can skew the results of any bifurcations analysis drastically. Case (a) represents the valid Floquet multipliers and cases (b), (c) and (d) show three possible cases where invalid Floquet multiplies can cause one to interpret the results as an entirely different bifurcation.

For example, consider the case where three successive periodic orbits, each one with two Floquet multipliers very close to unit circle at +1 (or Floquet exponents with very small real parts), which represents a very frequent scenario for oscillatory systems. Figure 4.8 illustrates that even if only one of the six Floquet multipliers is not accurate the resultant bifurcation analysis can be skewed dramatically. To explain this, suppose that the configuration in (a) represents the accurate Floquet multipliers where they march towards +1 on the unit circle from opposite directions. This configuration reveals that, although all three periodic orbits are unstable, no bifurcation happens. However, if the Floquet multipliers of the third periodic solution were to be invalid, as shown in configuration (b), the emergent sequence would suggest a fold (turning-point or saddle-node) bifurcation. In case (c), where the Floquet multipliers of the second periodic solution are inaccurate, it implies that two fold bifurcations (in two dimensions) have happened.

Case (d) poses a more challenging problem which unfortunately can happen very frequently in oscillatory systems. The challenge is not only because of invalid Floquet multipliers but also because numerical algorithms inevitably return non-zero values even for trivially zero (imaginary parts of) Floquet multipliers. When this happens, as case in (d) exemplifies one possibility, bifurcation analysis suggests that the system has two torus bifurcations in two dimensions. This issue is further discussed in the next section.
Figure 4.9: Floquet analysis of a small section of the second branch of periodic orbits in the linear region reveals a period-doubling bifurcation as the period of the periodic solutions decreases. Top: Frequency-Energy and also Poincare map representations of the mentioned section. Bottom-Right: Real parts of valid Floquet exponents associated with the periodic orbits on the section. All four (real parts of) Floquet exponents start at zero line. However at the bifurcation point, two exponent bifurcated from the zero line indicating that the following periodic orbits are unstable. Bottom-Left: Real Floquet multipliers leave the unit circle at $-1$ which indicates a period-doubling bifurcation.
Figures 4.10 and 4.9 showcase two examples of period-doubling and torus bifurcations that would have been mistaken by different types of bifurcations if one used invalid Floquet multipliers/exponents. In fact, in the case shown in Fig. 4.10, the small closed path (the askew ellipses in the magnified views at the bottom) of Floquet multipliers attached to the unit circle would have not appeared if smaller Floquet exponents, i.e. $< 5e - 4$, were disregarded or invalid exponents were used. Furthermore, as mentioned in the previous section, the system does not have to be in a (strongly) nonlinear region to include cases of bifurcations that would consequently cause some unexpected phenomena such as jump or period lengthening.

Figure 4.9 is also an example of a period-doubling bifurcation in a (fairly) linear region. As it will be discussed in the next section, there exist more bifurcations in the linear region but due to limitations imposed by the periodic solver, the integrator and ultimately by the machine’s precision, these bifurcations cannot be characterized.

Next, Figure 4.11 compares the results of Floquet analysis for another branch of 6735 periodic orbits, one with predictive validation algorithm presented here, and the other one with a fixed set of tolerances $\{h = 1e - 7 \times [1 1 1 1]^T, \epsilon = 1e - 6\}$, resulting in a similar total computation time of 1421 s and 1411 s. In the latter case, accurate bifurcation analysis is near impossible as a result of massive fluctuations around the zero surface.

### 4.3.3 Limitations

Like any other numerical algorithm, the ultimate accuracy of the proposed algorithm, which manifests itself in very small Floquet exponents, is firmly bounded by the machine’s precision. Specifically, this
Figure 4.10: Floquet analysis of a small section of the first branch of periodic orbits reveals a torus bifurcation. Top-Left: Frequency-Energy and also Poincare map representations of the mentioned section. Top-Right: Valid Floquet exponents associated with the periodic orbits on the section. Bottom: Complex Floquet multipliers leave the unit circle which indicates a torus bifurcation.
limitation casts an inevitable and unrecoverable uncertainty over the results of bifurcation analysis that rely on the accuracy of Floquet exponents. For instance, using a MATLAB ode function as the integrator and a Shooting algorithm as the periodic solver, there will always be a range of Floquet exponents, usually defined by $|\Re\{\mu_i\}| < 1e^{-12}$, that can not be validated. This region includes two types of exponents; a) invalid exponents that can be validated if higher precisions are available and b) invalid exponents that can never be validated because they represents (analytical) zero exponents.

Now consider the example depicted in Fig. 4.12. The difference between case (a) that implies a global bifurcation at a periodic solution with period $T_a$, cases (b) that indicates no bifurcation or case (c) that shows a bifurcation at $T_c$, can only be determined through separation of the mentioned two types of invalid Floquet exponents.

Moreover, cases such as case (d), although inaccurate, can hint at undetected bifurcations and consequently new branches of periodic solutions. The existence of such cases can be readily confirmed.
by comparing the number of (closed) branches of periodic orbits and number of detected bifurcation along the main two branches of periodic orbits. For example, in the case of the second branch of periodic solutions (in the positive $x_1(0) \geq 0$ half-plane), shown in Fig. 4.4, only five bifurcations have been detected\(^6\). But, one can readily observe from the (cross section of the) Poincare map in Fig. 4.5 that, as the period of the solutions along this branch decreases, at least twenty-four new branches of periodic solutions separate from it, which according to the bifurcation theory can only be the result of bifurcations of solutions on this branch. This amplifies the significance of methods such as the ICM or MMC algorithms in Sec. 3, which can be used to search for periodic orbits anywhere in the state space and without very accurate initial guesses. In a case such as this accurate bifurcation analysis is not possible due to the machine’s precision, but the bifurcations can be inferred from these branches in the Poincare map.

### 4.4 Mechanisms of Losing Stability (Cases of Local and Global Bifurcations)

Previous sections discussed how the stability of a periodic orbit manifests itself through Floquet multipliers (or exponents). This section explains the relationship between the Floquet multipliers (or exponents) and the orientability of the local invariant manifolds of the periodic orbits. More specifically, it explains some of the mentioned relations based on mechanisms through which fixed points or periodic orbits lose stability and as a result new periodic orbits come to exist. These mechanisms describe the change in the structure of the geometry of the periodic orbits and their local invariant manifolds as a function of the change in their period. In this regard two mechanisms explaining two types of bifurcations, with an eminent role in the system in Eq. (2.5), are presented. Note that the system in Eq. (2.5) exhibits a very rich set of different types of bifurcations, including some special cases. However, discussing all

\(^6\)That includes the one double-Hopf bifurcation at the center.
such cases of bifurcations is beyond the scope of this thesis. For more details on different types of bifurcation, specially in $\mathbb{R}^4$, readers are referred to [89, 63, 121, 16].

4.4.0.1 Bifurcations of Periodic Orbits from a Fixed Point

**Hopf-Hopf Bifurcation**- An equilibrium point is referred to as a Hopf-Hopf point (or double Hopf point) if, at the mentioned point; first, the system has only two pairs of purely imaginary eigenvalues, i.e. $\pm i\omega_1, \pm i\omega_2$ and, second, if the change in the imaginary part of the Floquet exponents with respect to the period is zero, or specifically, the transversality (generosity) condition (see [89, 63, 16])

$$\frac{d}{dT}\mu_i = 0$$

(4.25)

is satisfied. This is a broad definition that also includes unusual Hopf points. In a four dimensional system, at a Hopf-Hopf point many cases of bifurcation can occur which are described in [63]. In the case of the system in Eq. (2.5), bifurcation of double Hopf points is the birthplace of four (half) branches of periodic orbits on two dimensional invariant linear (and later curvilinear) planes. These branches of periodic orbits include linear modes of vibration and their nonlinear continuation. In this sense, one can state that the Hopf-Hopf bifurcation of the fixed point of the system (2.5) is the birthplace of the modes of vibration in that neighborhood.
Figure 4.13: Top: Real parts of Floquet exponents calculated for ICM I and ICM II and shown on the cross-section of the Poincare map described in Section 2. The period $T$ is defined as an descending variable along theses branches as they move away from the fixed point at the origin. The transversality (generosity) condition, i.e. $\frac{d\mu_i}{dT} = 0$, is satisfied at the fixed point of the system (2.5) at the origin. Therefore, the fixed point at the origin is a double Hopf point for the system (2.5). Bottom: Imaginary parts of Floquet exponents calculated for ICM I and ICM II.

4.4.1 Bifurcations of Periodic Orbits from a Periodic Orbit

Neimark-Sacker Bifurcation- A Neimark-Sacker (Torus) bifurcation\(^7\) is characterized by a pair of complex conjugate multipliers crossing the unit circle at

$$\rho = e^{\pm i\phi}, \phi \neq 0, \pi.$$  \hspace{1cm} (4.26)

\(^7\)It is sometimes called Hopf bifurcation of periodic orbits, secondary Hopf bifurcation or generalized Hopf bifurcation.
The bifurcation into a torus only happens in three or higher dimensional systems. In this case, a periodic orbit, that has lost its stability, spirals around a torus. This is schematically shown for a 3 dimensional case in Fig 4.14. Moreover, as shown in Fig. 4.14, in case of a torus bifurcation, the fixed point of Poincare map becomes unstable and generates limit cycles.

Figure 4.14: Left: A schematic illustration of torus bifurcation. A stable periodic orbit becomes unstable surrounded by a stable 2D torus. Right: Poincare map (shown with black dots) of a periodic orbit (shown in solid red) initiated at (0.9061 -1.6894, 0, 0). Appearance of limit cycles in the Poincare map reveals a case of torus bifurcation.

Numerous cases of torus bifurcations occur in the system (2.5). This can be observed in Figure 4.15 which shows the Floquet multipliers of two branches of periodic orbits. For example, in ICM I-5,II-9, the type-4 periodic orbits (in Table 1) bifurcate from the periodic orbit with multipliers shown in solid black circles through torus bifurcation. Both pairs of multipliers move outside the unit circle. The type-8 periodic orbits shown in red solid circles are the products of torus bifurcation of the periodic orbit with multipliers shown in solid green circles. In addition, in ICM II-56, the type-9 periodic orbits are created through torus bifurcation of the periodic orbit with multipliers shown in solid black circles. A pair of multipliers moves inside the unit circle while the other one moves outside.
4.5 Orientability of Invariant Manifolds

To understand the dynamics of conservative vector fields, one typically looks for attractors (stable limit cycles or periodic orbits), their basin of attraction and, in particular, the boundaries of these attracting domains. Invariant manifolds often act as basin boundaries, and it is important to know where they are located and what they look like [71, 89]. These manifolds are sometimes called separating manifolds, as they separate the state space into two invariant regions. Orbits starting on one side of the manifold are confined to this region since they cannot pass through an invariant manifold. In particular, a separating manifold must have two sides, namely an inside and an outside. However, there are two-dimensional

Figure 4.15: Left: Real part of Floquet multipliers calculated for ICM I-5,II-9 (Top) and ICM II-56 (Bottom). Right: Floquet multipliers of ICM I-5,II-9 (Top) and ICM II-56 (Bottom) in the neighborhood of the unit circle.
invariant manifolds that do not have two sides. Such manifolds are called nonorientable manifolds.

![Figure 4.16: Left: The Mobius strip is a nonorientable surface. Middle: A twisted strip represents an orientable surface. Right: Torus is also an orientable surface.](image)

The classical example of a nonorientable manifold is the Mobius strip. A Mobius strip can be easily made by wrapping a strip of paper around a circle and gluing its ends together after giving it half of a twist (see Fig. 4.16). This surface only has one side since any trajectory that starts on the top surface will end up on the opposite surface after one cycle. In other words, orbits can advance through this type of surface without crossing it. Two examples of orientable surfaces, i.e. a twisted strip and a torus, are also shown in Fig. 4.16. Orbits that starts on one side of the twisted strip or inside of a torus can not move through these surfaces without crossing them.

The Mobius strip is the only two dimensional non-orientable manifold that can be embedded in $\mathbb{R}^3$ without self-intersection. For example, for the system in 2.5, the local invariant manifolds of the period lengthened periodic orbits form complicated manifolds that intersect themselves. A few of these manifolds are shown in Fig. 4.17. However these manifolds are nonorientable and have very similar structures to that of the Mobius strip. The complexity of these manifolds is caused by the shape of their periodic orbits (mode shapes). In other words, to construct these manifolds, like the Mobius strip, one can start with a strip of paper but, in this case, the strip needs to be wrapped along the mode shape of the periodic orbit instead of a circle. Also, its ends now need to be glued together after an odd number of half twists.
Figure 4.17: Examples of nonorientable manifolds. Left: A local ICM representing a twisted strip with \((5 + 8) \times \frac{1}{2}\) (thirteen half) twists around a periodic orbit, shown with a solid black line, with 5:8 resonance. Right: A local ICM representing a twisted strip with \((16 + 1) \frac{1}{2}\) (seventeen half) twists.

The system (2.5) also has several periodic orbits along orientable manifolds. For example, Fig. 4.18 represents three cases of twisted strips with, respectively, one, four and six full twists and a case of a torus local ICM.

These local invariant manifolds are key building blocks in shaping the orbits of nonlinear conservative systems, since, they can be seen as the gates and walls of the state space that the orbits of these systems can move through and cannot cross, respectively. In this regard, one needs a quantitative tool to determine the orientability (or nonorientability) of these local ICMs. This can be done through the stability analysis of their corresponding periodic orbits.
4.6 Determining the Orientability of Invariant Manifolds

This section explains a quantitative approach that can determine the orientability property of local invariant manifolds using the Floquet multipliers and exponents of the corresponding periodic orbit about which the manifold was created. In this approach, first, periodic orbits are categorized based on their Floquet exponents as shown in Table 4.2. Then, the orientability of both the local stable and unstable manifolds, if they exist, are determined using Floquet multipliers. This orientability property is the key to understanding whether a periodic orbit can be reached by the system from a particular region in the state space, and hence whether the system is likely to jump or remain on a particular solution branch. For example, in the damped system studied herein, since the system is losing energy, it will always approach any periodic orbit, on its path, from its local stable manifold. Therefore, if the stable manifold of a periodic orbit is orientable, the periodic orbit in question will not affect the path of the solution of the system. In [7], these periodic orbits are referred to as isolated periodic orbits. Next, the orientability of local center manifold is determined using Floquet multipliers. However, local
center manifold do not always exist. In such cases, one can assign the orientability of the local ICM to the periodic orbit. That is because, the orientability of the local center manifold and the local ICM should agree at all times since ICMs are the loci of all periodic orbits of the system. In other words, in such cases, periodic orbits adopt the orientability of the center manifolds of periodic orbits in their neighborhood on the local ICM.

The basis in identifying the orientability property of local stable and unstable manifolds of periodic orbits is the fact that Floquet multipliers of a periodic orbit are the same as eigenvalues of its Poincare map [89, 70, 121]. In [88, 87, 104] it was shown that a local manifold is orientable if all of the eigenvalues of its Poincare map are real and positive. This was done using the fact that the Poincare map is an orientation preserving map.

The orientability of a local center manifold is based on the type of bifurcation that has generated it. It was revealed in [7] that the system (2.5) exhibits, mostly, three types of bifurcations, namely, Hopf-Hopf bifurcations, Period-Doubling bifurcations and Andronov-Hopf (or Torus) bifurcations. One can determine which type of bifurcation one has based on the Floquet multipliers of the periodic orbit, as discussed in [7, 89, 79, 15]. In this regard, the local center manifold of a periodic orbit is:

- an orientable curvilinear plane if it is originated from a Hopf-Hopf bifurcation,
- a twisted strip with $k$ half-twists (orientable for even $k$ and nonorientable for odd $k$) if it is originated from a Period-Doubling bifurcation, or
- an orientable torus if it is originated from an Andronov-Hopf (torus) bifurcation.

In cases where the local center manifolds do not exist, such as cases of periodic orbits with Period-Doubling bifurcations, the local ICMs can be used to determine the orientability property of the periodic orbits. In this regard, since, all coordinates of a periodic solutions are also periodic, one can assume an analytical representation of the ICMs [5] of the system 2.5 as

---

8 Poincare map is a rotation matrix with unity determinant.
\[ x_1 = \sum A_m^{(1)} \cos m\phi_1 = \cos \phi_1 \left( A_1^{(1)}(T) + \sum_{m=2} A_m^{(1)} \cos m\phi_1 \right) \]
\[ H_{x_1} = \sum A_m^{(1)} \sin m\phi = \sin \phi_1 \left( A_1^{(1)} + \sum_{m=2} A_m^{(1)} \sin m\phi_1 \right) \]
\[ x_2 = \sum A_m^{(2)} \cos m\phi_2 = \cos \phi_2 \left( A_1^{(2)} + \sum_{m=2} A_m^{(2)} \cos m\phi_2 \right) \]

(4.27)

where \( \phi_i \)'s, in general, are

\[ \phi_1 = q\phi_0 \]
\[ \phi_2 = p\phi_0 \]

(4.28)

and \( \phi_0 = \frac{2\pi}{T} t \) and \( T \) represents the period of the periodic orbits. Equation (4.27), in general\(^9\), defines a twisted strip with \( p+q \) half-twists (see ruled surfaces in [48, 59]). In this sense, following the discussed in Section 3, if \( p+q \) is odd, then (4.27) defines a nonorientable surface. On the other hand, the surface is orientable if \( p+q \) is even.

To this end, nine types of periodic orbits are identified and the orientability of their local invariant manifolds (including the local ICM) are summarized in Table 1. It is importance to mention that Table 1 applies only to a 2 DOF conservative system in question and is far from providing a complete list for a general dynamical system in \( \mathbb{R}^4 \).

\(^9\)In some special cases, such as where \( \phi_2 = \frac{(2k+1)\pi}{2} - \frac{l}{2} \phi_1, \) \( k, l \in \mathbb{N} \), Eq. (4.27) generates Tori.
Table 4.2: Nine types of periodic orbits are identified for the system (2.5) based on their Floquet multiplier and exponents. Floquet multipliers and exponents are denoted by $\rho$ and $\mu$ respectively.
4.7 Cases of Jump Phenomena in the Free Response of the Non-conservative System

In this section, the non-conservative system shown in Fig. 2.2 is considered herein with non-zero damping coefficients with the same underlying conservative system as the system in (4.19). The EOM of this system can be written as

\[
\begin{bmatrix}
\ddot{x}_1 \\
\ddot{x}_2 \\
\ddot{y}_1 \\
\ddot{y}_2
\end{bmatrix} = - \begin{bmatrix}
2 & -1 \\
-1 & 2
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} - C \begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{y}_1 \\
\dot{y}_2
\end{bmatrix} - \begin{bmatrix}
0.5x_1^3 \\
0
\end{bmatrix},
\]

(4.29)
or equivalently

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{y}_1 \\
\dot{y}_2
\end{bmatrix} = \begin{bmatrix}
0_{2\times2} & I_{2\times2} \\
-2 & 1
\end{bmatrix} - C \begin{bmatrix}
x_1 \\
x_2 \\
y_1 \\
y_2
\end{bmatrix} + \begin{bmatrix}
0 \\
0 \\
-0.5x_1^3 \\
0
\end{bmatrix},
\]

(4.30)
where the damping matrix \( C \) is defined by

\[
\begin{bmatrix}
0_{2\times2} & 0_{2\times2} \\
0_{2\times2} & -C
\end{bmatrix} = V^T \begin{bmatrix}
0_{2\times2} & 0_{2\times2} \\
0_{2\times2} & 2\zeta_1\omega_n \ 0 \\
0 & 0 \\
0 & 2\zeta_2\omega_n
\end{bmatrix} V
\]

(4.31)
in which, \((i\omega, V)\), define the linearized modes of the conservative part of the system in (4.29). This way, one can readily find the damping matrix

\[
C = V^T \begin{bmatrix}
2\zeta_1\omega_n1 & 0 \\
0 & 2\zeta_2\omega_n2
\end{bmatrix} V = 2 \begin{bmatrix}
-\frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \\
-\frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2}
\end{bmatrix}^T \begin{bmatrix}
\zeta_1 & 0 \\
0 & \sqrt{3}\zeta_2
\end{bmatrix} \begin{bmatrix}
-\frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \\
-\frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2}
\end{bmatrix}
\]

(4.32)
that should be used to give two desired modal damping ratios in the system (4.29).

This system is initiated at an initial condition that is on one of the periodic orbits on ICM I or ICM II, but with a small deviation \(\delta\). The time evolution of the solution is then calculated using the EOM
defined in (4.29) for different values of $\zeta$ and $\delta$. Then, the short time Fourier transform is used to obtain the spectrogram of the solution. The dominant frequency at each time instant is then extracted from the spectrogram. Figures 4.21-4.23 show the results of a few of these simulations in which jump phenomena are observed as the system loses energy along different invariant manifolds of the underlying conservative system. Two (out of many possible) scenarios are outlined next to explain these jumps as the interaction between invariant manifolds of the system.

Figure 4.19: Time-frequency history of the solution of the system, initiated on the ICM II, is calculated using short-window Fourier transform.

In order to explain the jumps in the dominant frequency of solution of the system, for example in the time-frequency plot shown in Fig.4.19, some background could be beneficial. First, the (scaled) frequencies of the periodic solutions of the system (4.19) are shown as functions of their their energy in Fig 4.20. The scaled frequency $f_s$ is defined as
\[
    f_s = \begin{cases} 
    f & \text{ICM I and ICM II} \\
    \frac{1}{f_{\text{max}} - f_{\text{min}}} [m(f - f_{\text{min}}) - n(f - f_{\text{max}})] f & \text{Otherwise}
    \end{cases}
\]

(4.33)

where \( f \) is the fundamental frequency of periodic orbits of the system (2.5), and \( m \) and \( n \) are the positive integer factors in the terms ICM I-m, II-n referring to an invariant manifold (ICM) that intersects with the ICM I and ICM II at periodic orbits that are cases of, respectively, \( m:1 \) (\( m \) to one) and \( n:1 \) resonances, i.e. \( f_1 = mf_2 \) and \( f_1 = nf_2 \). It is important to notice that the frequency \( f_s \) represents the fundamental frequency only for the ICM I and ICM II. For all of the other ICMs, \( f_s \) represents a scaled version of their fundamental frequencies, where, the scaling factor is a positive integer only at the intersection of these ICMs with the ICM I and ICM II. This scaling is done to show all of the branches of the nonlinear modes in a region between the two branches defined by ICM I and ICM II.

Figure 4.20: Scaled frequencies of periodic orbits nonlinear modes on ICMs are shown as functions of their energies for the same ICMs. Scaled frequencies are calculated using Eq. (4.33). One group of ICMs, shown in cyan, start from the ICM II and land on ICM II while the other group of ICMs, shown in black, start from the ICM I (ICM II) and land on ICM II (ICM I).

Second, moving along an invariant manifold of the system, in this case the local center manifold or local ICM, the solution of the system may change its path from one manifold to another manifold.
that comes to contact with it. In this scenario, in order for such a change to happen, two ingredients are required. First, the two mentioned manifolds should be tangent to each other at the state where the solution changes its course from one to another. This tangency or non-transversality can be captured by the resonance condition or equivalently in the transversality condition. At the resonance condition, the Jacobian of the linearized system around the equilibrium becomes singular, indicating that its manifold may have more than one tangent plane at that equilibrium. The non-transversality condition described in Eq. (4.25) expresses the same concept in terms of (real part of) Floquet exponents and the change in the stability of the equilibrium. Second, the solution of the system needs to be perturbed (or be in a perturbed state) while passing through the neighborhood of the non-transversal intersection. In this regard, the initial perturbation $\delta$ is imposed to the solution of the system on the local ICMs. Note that when the manifold (NNM branch) is stable, the deviation $\delta$ will gradually decay due to damping until it is no longer strong enough to cause a jump in the path of the solution.

Jump phenomena may also occur when the solution of the system changes its path because of the orientability properties of the manifolds on its path. The manifold described by one of the NNM branches is typically a curvilinear plane or a twisted strip. It may be intersected by a variety of other manifolds. For example, suppose that the second manifold is a closed orientable manifold such as a torus. In this case, the solution of the system cannot lose energy by orbiting outside of the torus and needs to move to the inside. That is not possible because torus defines an orientable manifold. In such cases, a jump to the next open manifold is likely to happen. On the other hand, twisted Mobius stripe and twisted strips are both open (nonorientable and orientable) manifolds where the solution of the system can lose energy while orbiting on one side of the manifold without having to cross it, and so, the response of the system is likely to stay on this type of manifold and not exhibit a jump.
Figure 4.21: Dominant frequency of the decaying solution of the system is overlaid on top of ones from identified ICMs of the underlying conservative system. Unstable periodic orbits are shown in red dots for five of the ICMs, i.e. ICM I, ICM II, ICM I-5,II13, ICM I-5,II14 and ICM I-5,II15. Jumps occur at six neighborhood labeled A to F.

Figure 4.21, shows an example where the solution was initiated on the ICM II with $\delta = 0.05\%$ deviation in its initial condition. The damping ratio was set to $\zeta_{1,2} = 1e^{-7}$. Each neighborhood of the points marked with circles (denoted by letters A to F) houses two consecutive jumps which will be explained next. Starting (almost) on ICM II, the solution of the system begins losing energy on one side of ICM II (an orientable Twisted strip) until it hits point A. Two consecutive jumps, one because of non-transversality and another because of an orientable torus happens in the neighborhood A. First, the solution jumps from ICM II to ICM I-5,II-15 because of non-transversal intersection of the ICM II and the ICM I-5,II-15 manifolds become tangent (15:1 resonance). However, this jump is not evident in the dominant frequency shown in Fig 4.21, because, the scaling function described in (4.33) is used to scale the dominant frequency. In the absence of this scaling function, the dominant frequency would be seen to jump to 1/15th of the ICM II frequency. Then, since the manifold ICM I-5,II-15, at the neighborhood of A is a torus, the solution jumps again to ICM I in the neighborhood B where ICM I is locally a orientable twisted Mobius strip. The solution continues to lose energy until it reaches the end
of the neighborhood B where the local ICM I is tangent to ICM II (3:1 resonance-the local ICM of this point is visualized in Fig (4.18)- Middle-Left). At this point, the solution jumps back on the ICM II and continues to lose energy on one side of its orientable twisted strip until it enters in the neighborhood C. Same set of jumps as in the neighborhood A happens in neighborhood C causing the solution to jump down on the ICM I which is locally an orientable twisted strip in the neighborhood D. Another sets of two consecutive jumps similar to neighborhoods A and C, send the solution back up to the ICM II (neighborhood E) and then down to ICM I (neighborhood F). However, at this point, the deviation δ has decayed to a level that is not enough to cause any other jumps at the upcoming points prone to jump at its path on ICM I.

Figure 4.22: Left: Poincare map sowing the ICMs I, II and I-5,II-14. Right: Magnified view showing the solution of the system at the neighborhood B and C highlighted by red arrows.

These jumps are examined in more detail in Fig. 4.22, which shows a cross section of a Poincare map of the system (defined in Section. 2) that shows several of the ICMs of the system in the neighborhoods B and C that were marked in Fig. 4.21. Moving along the the ICM I, the solution reaches the non-transversal intersection between the ICM I and ICM II at the point $B_1$ in the neighborhood B. At this point the deviation δ causes the solution to jump from ICM I to ICM II and continue on it until it reaches the point $C_1$ in the neighborhood C. At this point the ICM II and ICM I-5-II-14 intersect non-transversaly
and the solution jumps onto the ICM I-5-II-14 and it continues on ICM II until point $C_2$. The point $C_2$ represents a periodic solution that orbits on the surface of a torus (its local ICM). The periodic orbit at $C_2$ is the product of the periodic orbit at $C_1$ after losing stability through torus bifurcation. At this point, since the solution cannot lose energy by orbiting outside of a torus and also cannot move through it, since the torus is orientable, it jumps again onto the ICM I in the neighborhood D.

In another case, shown in Fig. 4.24, the solution, which was initiated on ICM I, reveals a similar set of jumps at the neighborhood B, C, and D and finally rest at ICM II. As explained earlier, damping also causes the deviation to decay as the solution advances. This effect is evident at the neighborhood F where the deviation has decayed to such an extent that it is not enough to cause another jump despite the existing potential for another set of jumps at its next encounters with non-transversal intersections. This effect can be further examined by recomputing the response with a damping ratio $\zeta_{1,2} = 1e - 6$, that is ten times larger than what was used in Fig. 4.21. To illustrate this effect, the dominant frequency of the solution of the system (for the same set of initial conditions and deviation) are obtained for damping ratios $\zeta_1 = \zeta_2 = \{5e - 7, 1e - 6, 12e - 6\}$. As shown in Fig. 4.23, increasing the damping ratio will cause the initial deviation to decay more quickly so that it falls below the level at which it is sufficient to cause a jump at a non-transversal intersection. At $\zeta = 12e - 6$, the initial deviation is damped enough before the solution reaches the first non-transversal intersection. It is worth mentioning that decreasing the initial deviation has the exact same effect as increasing the damping ration on the behavior of the solution at non-transversal intersections on its path.
Figure 4.23: Dominant frequency of the solution of the damped system, shown with a solid green line, obtained for damping ratios $\zeta = \{5e^{-7}, 1e^{-6}, 12e^{-6}\}$. Increasing the damping ratio, while the initial deviation was held fixed, has caused the initial deviation to decay quickly to below the level at which it is large enough to perturb the solution at its first encounter with the non-transversal intersection at neighborhood A and at all subsequent intersections. It is also interesting to note that the response continues to decay along the ICM II even though many of the periodic orbits are unstable for energies below $10^1$. 
Figure 4.24: Dominant frequency of the decaying solution of the system initial (almost) on the ICM I with $\delta = 0.05\%$ and $\zeta = 1e^{-6}$. At three neighborhoods B, C, and D the deviation of the solution from the periodic orbits of the underlying conservative system forces the solution to move form one ICM to another one at their non-transversal intersections causing the frequency of the solution to jump. It's worth noting that although the solution was initiated on the ICM I, it reaches the origin through losing energy on the ICM II.
Chapter 5

Connecting Functions of a Class of Nonlinear Systems

5.1 Introduction

Homogeneous linear systems\(^1\), by definition, admit superposition since it combines two properties of such systems, i.e. homogeneity and additivity. Superposition can be used to define the relationship between the solutions of linear systems. In particular, for the homogeneous linear system

\[
\dot{x} = Ax,
\]

where \(x \in \mathbb{R}^n\) and \(A\) is a \(n \times n\) real matrix, one can state that because this linear system accepts superposition, then any function

\[
F(\bar{x}_1, \ldots, \bar{x}_k) = \sum_{i=1}^{k} \alpha_i \bar{x}_i, \quad \alpha_k \in \mathbb{R}, \ k \in \mathbb{N},
\]

\(^1\)In this study, the term system refers to any continuous, autonomous system of ordinary differential equations unless mentioned otherwise.
where $\bar{x}_i$ satisfies Eq. (5.1), is a solution of the system. In other words, superposition defines a linear function, i.e. $\mathbf{F} : \mathbb{R}^n \times \ldots \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, that connects a new solution of the system to a set of its known solutions, i.e. $\bar{x}_k \in \mathbb{R}^n$. Therefore, as proposed in [76], one can call the function $\mathbf{F}$ a linear connecting function.

The extension of this concept to nonlinear systems, i.e finding connection functions of nonlinear systems, as a tool for finding new solutions of such systems is the main focus of this study. More specifically, for the nonlinear system

$$\ddot{x} = f(x),$$

(5.3)

where $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a $C^r$, $r \geq 1$ continuous nonlinear vector function and $S$ and $\bar{S} \subset S$ are respectively the set of its entire solutions and a set of its known solutions, this study investigates the existence, general forms and methods of identifications of functions $\mathbf{F} : \bar{S} \rightarrow S$ in order to find new solutions of the nonlinear system in Eq. (5.3). Such an extension is faced with a few challenges. In order to point out these challenges, the next section briefly explains how superposition or linear connecting functions has been used to serve a similar purpose for linear systems.

5.1.1 Superposition: A Tool for Solving Linear Systems

Consider the linear (connecting) function defined by superposition in Eq. (5.2). One can use this connecting function to generate a new solution of the system, if a set of its solutions is known. In other words, it can be regarded as a tool to solve linear systems for new solutions. For example, consider the following linear system of ODEs which describes the governing equations of motion for a two degrees-of-freedom (DOF) oscillator shown in Fig. 5.1.

$$
\begin{bmatrix}
\ddot{x}_1 \\
\ddot{x}_2
\end{bmatrix} =
\begin{bmatrix}
-2 & 1 \\
1 & -2
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix},
$$

(5.4)
Figure 5.1: A 2-DOF linear oscillatory system with $k_1 = k_2 = k_3 = 1$ and $m_1 = m_2 = 1$.

This system has two sets of distinct linear eigensolutions (modes) described by

$$
\bar{S}_1 = \{ \bar{x}_{1LM} = a_1 \bar{x}_{1LNM} = a_1 \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \cos(t) | a_1 \in \mathbb{R} \} \\
\bar{S}_2 = \{ \bar{x}_{2LM} = a_2 \bar{x}_{2LNM} = a_2 \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \cos(\sqrt{3}t) | a_2 \in \mathbb{R} \}.
$$

where $\bar{x}_{1LNM}$ and $\bar{x}_{2LNM}$ linear normal modes of the system. Solutions from these two sets can be used in the linear (connecting) function in Eq. (5.2) to find new a) arbitrary solutions, where the new solutions are unconstrained, i.e. the unknown parameters $\alpha_i$ in (5.2) can span the entire set of real numbers or b) fixed solutions, i.e. specific solutions of interest, where the new solution is constrained to pass through a fixed point in the state space.

Finding arbitrary solution is possible for linear systems because of two facts: first, superposition provides a form for the connecting function $F$ and second, it guarantees that $F$ will always be a solution of the system. In other words, superposition combines two conditions that must be satisfied for any function $F(\bar{x}_1, ..., \bar{x}_k)$ to be a solution of the system; a) it must be of the form expressed in Eq. (5.2) (a necessary condition) and b) it must satisfy the governing differential equations of the system in Eq. (5.1) (a sufficient condition). In this way, for any set of $k$ solutions from $\bar{S}_1$ and/or $\bar{S}_2$ and any combinations of $(\alpha_1, ..., \alpha_k)$, $\alpha_i \in \mathbb{R}$, the connecting function $F$ generates a new solution of the system. As a result, one can find infinitely many solutions of the system without solving its governing differential equations.
Figure 5.2: Connecting functions can be used to find the solution of the system at an arbitrary point by using a set of its known solutions.

In order to find new fixed (specific) solutions, given a set of known solutions $\bar{x}_i$, one simply needs to identify the unknown parameters $\alpha_i$. Conversely, one could fix the parameters $\alpha_i$ and then search for a scaled set of eigensolutions $\bar{x}_i$. To explain this, let’s start with a schematic view of the known solutions from $\tilde{S}_1$ and $\tilde{S}_2$. In this regard, Fig. 5.2 shows the known solutions $\bar{x}_1^{LM}$ and $\bar{x}_2^{LM}$ in the $x_1 - x_2 - t$ frame. As it’s evident in this figure, the intersection of the known solutions $\bar{x}_1^{LM}$, $\bar{x}_2^{LM}$, $\bar{x}_1^{LN}$, $\bar{x}_2^{LN}$ and the plane $\Omega = \{(x_1, x_2, t)|t = 0\}$ form respectively two lines for the linear modes and two points for the linear normal modes.

Now, suppose one is seeking the solution of the system that passes through an arbitrary point on $\Omega$, denoted herein by $(u, v = 0)^2$, without solving the differential equations in Eq.(5.4). In this case, neither the set of coefficients $\alpha_i$, $i = 1, ..., k$ nor the known solutions $x_k$ can be arbitrary. That is because, to obtain the desired solution, the coefficients $\alpha_i$ must be chosen so that the connecting function gives the desired response at the initial time. In other words,

$$\sum_{i=1}^{k} \alpha_i \bar{x}_i(0) = u, \quad \sum_{i=1}^{k} \alpha_i \dot{\bar{x}}_i(0) = v.$$  

(5.6)

Throughout this study, without loss of generality, and in order to simplify visualization of the examples, $v$ remains identical to the null vector.
Immediately, one may recall from linear algebra that, unless \( u \) is on one of the lines representing \( \bar{S}_1 \) or \( \bar{S}_2 \) in Fig. 5.2, the sum above must include at least two solutions, i.e. one from each \( \bar{S}_1 \) and \( \bar{S}_2 \), in order to yield a system of algebraic equation of \( \alpha_i, i = 1, \ldots, k \) with a solution. Moreover, the number of solutions, i.e. \( k \), cannot be greater than the two, otherwise, the algebraic system of equation in Eq. (5.6) becomes under-determined. Fortunately, for a homogenous system, normalization of all the eigensolutions in \( \bar{S}_1 \) and \( \bar{S}_2 \) leads to two unique eigensolutions, namely, linear normal modes \( \bar{x}^{LNM}_1 \) and \( \bar{x}^{LNM}_2 \). These two normal eigensolutions are represented in Fig. 5.2 by two unit vectors on plane \( \Omega \) along the two lines representing solutions from \( S_1 \) and \( S_2 \). Now, we can rewrite the linear algebraic system in Eq. (5.6) as

\[
\alpha_1 \bar{x}^{LNM}_1(0) + \alpha_2 \bar{x}^{LNM}_2(0) = u
\]  

(5.7)

which is balanced and guaranteed to always have a solution\(^3\). The above system can be even further simplified if the linear normal modes \( \bar{x}^{LNM}_1 \) and \( \bar{x}^{LNM}_2 \) are orthogonal, in which case, the solution of the above system can be readily obtained as

\[
\alpha_1 = \langle u, \bar{x}^{LNM}_1 \rangle, \alpha_2 = \langle u, \bar{x}^{LNM}_2 \rangle,
\]

(5.8)

where \( \langle, \rangle \) denotes the inner product.

To this end, one can conclude that the linear connecting function defined by superposition in Eq. 5.2, makes it possible to replace the problem of solving the linear system of ODE’s in Eq. (5.4) (for example by numerical integration) by solving a linear system of algebraic equations in Eq. (5.7).

\(^3\)The second sets of equations, i.e. \( \alpha_1 \bar{x}^{LNM}_1(0) + \alpha_2 \bar{x}^{LNM}_2(0) = v \), are true for all \( \alpha_1, \alpha_2 \in \mathbb{R} \).
5.1.2 Superposition and the General Solution of Linear Systems

General solution can be defined as a connecting function $F$ of a finite set of known solutions $\{\bar{x}_1, \ldots, \bar{x}_k\}$ that can span the entire set of solutions of the system. In other words, if $S$ is the set of all solutions of the system, then the general solution is a surjective function $F : \{\bar{x}_1, \ldots, \bar{x}_k\} \rightarrow S$. Therefore, in order to find the general solution of a system, one needs to find the mentioned finite set of solutions in addition to the corresponding connecting function. For linear systems, this leads to an eigenvalue problem that returns a set of eigensolutions of the system. In other words, the (infinite) set $\bar{S}$ which encompasses infinitely many periodic solutions of the system also known as linear modes. However, the homogeneity property of linear systems allows for normalization of the solutions in the set $\bar{S}$ defined by

$$N(\bar{S}) = \{ \bar{x} \mid \bar{x} \in \bar{S}, \langle \bar{x}, \bar{x} \rangle = \int_0^T \bar{x} \bar{x}^T dt \}$$  \hspace{1cm} (5.9)$$

which reduces the infinite set $\bar{S}$ to the finite set $N(\bar{S})$. 

Figure 5.3: Left: Linear connecting functions can replace (numerical) integration by solving a linear system of algebraic equations in solving a linear system of ODE’s. Right: Nonlinear connecting functions can play the same role for nonlinear systems. This will be discussed thoroughly in the next sections.
with as many members as the size of the linear system. The resultant normal eigensolutions are also known as linear normal modes. If these normal eigensolutions of the system are all distinct, i.e. linearly independent, they can be used as the aforementioned finite set for the linear connecting function in Eq.(5.2) to span the entire set of solutions of the system.

5.1.3 Nonlinear General Solutions: Challenges and Attempts

Note that, without the homogeneity property, the normalization in Eq. (5.9) would return yet another infinite set. This point is particularly important, because for the nonlinear system in Eq. (5.3), the set of all periodic solutions is infinite and its size cannot be reduced simply by normalizing. Therefore, the set of eigensolutions (periodic solutions) of the nonlinear system (5.3), sometimes called nonlinear modes, remains an infinite set. For example, consider the same simple 2-DOF system with one additional nonlinear element (a cubic spring attached to mass 1) described by

\[
\begin{bmatrix}
\ddot{x}_1 \\
\ddot{x}_2
\end{bmatrix} = 
\begin{bmatrix}
-2 & 1 \\
1 & -2
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} - 
\begin{bmatrix}
0.5x_1^3 \\
0
\end{bmatrix}.
\] (5.11)

Figure 5.4 shows how the number of branches of periodic solutions of a nonlinear system can be dramatically larger than of its underlying linear system. These branches of periodic solutions were identified using the algorithms proposed in [5, 105, 8] and mentioned in Chapter 3.

Perhaps, the most important result of this shortcoming emerged in Lie’s work [92] in 1893 where he investigated the existence of a general solution for nonlinear systems. He was able to prove an important theorem connecting Lie algebras and a nonlinear connecting function for solutions of a class of non-autonomous systems of nonlinear ordinary differential equations (ODE) [92]. A complete proof
of Lie’s theorem on ordinary differential equations that admit a nonlinear connecting function is given in [32]. According to Lie’s theorem the only system in the form of Eq. (5.3) for which there exists a general solution is, up to a change of dependent variable, the Riccati equation[^5][92, 3, 4, 58, 32].

Consequently, instead of the general solution of the system, many attempted to provide some alternative techniques, inspired by superposition, in order to quantitatively analyze the nonlinear system (5.3). For example a superposition-like technique was used in [17] to approximate the frequency of the periodic oscillations of the nonlinear system (5.3) where the functions in \( f(x) \) were considered as superpositions of several force fields, each defined by one of the terms of the Taylor series expansion of \( f(x) \). In [18], nonlinear modes of a coupled nonlinear system were expressed as superpositions of the LNMs of each of the homogenous system whose force fields are superimposed to yield the force field of the original system. Inspired by linear modal analysis, Shaw and Pierre suggested in [122, 123] that any solution of the system (5.3) might be constructed from a linear combination of nonlinear modes on invariant manifolds of the system which suggested that the governing equations of such invariant manifolds may act as a nonlinear normalization definition and reduce the set \( \mathcal{S} \) to a finite set (This assumption is further examined in Appendix 10). Nevertheless, as it was predicted in [138, 118, 137] and later shown in [5], nonlinear systems can still posses infinitely many invariant manifolds.

However, note that the lack of a general solution does not imply that no general form of connecting functions exists[^6]. In fact, Lie’s theorem only imposes limitations on what can be the domain and the co-domain of a connecting function of a nonlinear system. In other words, Lie’s theorem does not disprove the existence of connecting function for nonlinear systems but only states that they cannot

[^4]: Lie’s theory was originally stated for non-autonomous systems.
[^5]: The Riccati equation can be expressed as \( \dot{x} = A(t)x + B(t)x^2 \).
[^6]: It is worth noting that the existence of a connecting function alone does not imply a general solution either. For instance, in the case of linear systems, superposition had been known long before the general solution of a linear system was found. This can be inferred by the fact that the concept of the eigenvalue, which plays a central role in finding the general solution, can be traced only as far back as in the works of Euler, Lagrange and Cauchy [83]. However, it is necessary to have a connecting function. In other words, the existence of a connecting function is necessary but not sufficient for the general solution of the system to exist.
Figure 5.4: (Repeated from Fig. 4.5 for convenience) Periodic solutions of a two DOF nonlinear oscillatory system, defined by with $k_1 = k_2 = k_3 = 1$, $m_1 = m_2 = 1$ and $k_{nl} = 0.5$, are shown in two views; (1) on the initial condition plane $\Omega$ and (2) on the energy-frequency plane. The periodic solutions of the underlying linear system, i.e. when $k_{nl} = 0$, match with the periodic solutions of the nonlinear system only in a very small region close to the equilibrium of the system at the origin.

have a finite domain and be surjective at the same time. This notion commenced a different approach, mostly in the mathematics community [76], that proposed to define and use connecting functions of (5.3) as a local general solution, i.e. connecting functions that can only span a subset of all solutions of the system. In this approach, pioneered by Jones and Spijker [76, 129], the superposition rule in (5.2), was considered as a special case, i.e. a linear connecting function.

For example, Oppenheim defined homomorphic systems [102, 103] as a class of nonlinear systems with the property that their response to the linear combination of inputs is determined by its response to each of inputs in the set in a nonlinear manner\footnote{For example consider a (known) response function $F$ with the property $F(\alpha_1 q_1 + \alpha_2 q_2) = F(q_1)^{\alpha_1} F(q_2)^{\alpha_2}$.}. However, Oppenheim’s definition (among other restrictions) was applicable only whenever the response of the system was known in closed form. Adopting the concepts of the connecting function from [76], Spijker proved that the only kind of connecting function...
(of the form (5.2)) possible for the system (5.3), subjected to some conditions, is a linear connecting function. This way, assuming that \( x_1, ..., x_k \in \mathbb{R}^n \), \( k \in \mathbb{N} \) are solutions of (5.3), according to Spijker\(^8\), a function \( F = F(q_1, ..., q_k) \) is a linear connecting function if \( F \) is also a solution of (5.3) and

\[
F = F(x_1, ..., x_k) = \gamma + \sum_{i=1}^{k} \alpha_i x_i,
\]

(5.12)

where \( \alpha_1, ..., \alpha_k \in \mathbb{R} \) and \( \gamma \in \mathbb{R}^n \) which do not depend on the solutions \( x_1, ..., x_k, k \in \mathbb{N} \) \([129]\). Spijker, further proved that all systems that can accept the above form of connecting functions must be linear.

Following Spijker’s results, it is clear that the quest for finding a general form for connecting functions of nonlinear systems needs to be pursued in a somewhat different path. In this work, we suggested that a viable solution can be obtained by deviating slightly from the connecting function definition in Eq. (5.12), may lead to such a path. In particular, we showed that such a change allows for nonlinear connecting functions and consequently a nonlinear (local) general solution to exist. Furthermore, since the nonlinear modes of the system 5.3 are analogous to linear modes of the system (5.1), in that they are both periodic, they can be used in the new definition of connecting functions as eigensolutions. In this thesis, we find a general form of (locally) surjective connecting functions that may accept infinitely many solutions.

### 5.2 Existence and Forms of Connecting Functions

In this section we study the general forms that connecting functions can accept for linear and nonlinear vector systems. Specifically, we seek to understand how certain restrictions, when applied to the domain and/or co-domain of a connecting function, affects its general form and its applicability. In this regard, it is crucial to, first, make the distinction between the forms of connecting functions and of vector fields that define the systems. In other words, as we will prove below, linearity or non-linearity of vector fields

\(^8\)Originally, Spijker presented the following for non-autonomous systems.
do not necessarily determine the linearity or non-linearity of their connecting functions. In fact, it is how we define the connecting functions that determines their general forms and consequently whether they can be used for finding solutions of a linear or nonlinear vector field.

This is a very significant distinction and perhaps the main contribution of this study, in that, it allows one to meet the desired/required engineering demands by an appropriate mathematical formulation. For example, in most engineering applications, it is the local behavior of the systems, whether it be defined by a certain frequency band or a range of energy or excitations, that is of interest. This, mathematically speaking, can be translated to restrictions on both the domain and the co-domain of the connecting functions, the very fact, that was not considered in the work of Lie on existence of global nonlinear superpositions or in the works of Spijker in finding the general forms of connecting functions of nonlinear systems. Particularly, the introduced flexibility in the definition of connecting functions allows one to reformulate the question of existence of connecting functions for nonlinear system, in a manner, that does not contradict the work of Lie which rejects the existence of global (and not local) nonlinear superposition or the findings of Spijker which rejects the existence of nonlinear connecting functions.

Note that the results of this section do not guarantee the existence of connecting functions but only provide a general form for them if they exist.

5.2.1 Linear Connecting Functions

A formulation similar to the one used by Spijker [129] is adopted herein to prove a similar result (which also agrees with Lie’s theorem) using the connecting function concept. In other words, if the domain and the co-domain of a connecting function are both set to span the entire set of the solutions, i.e. the connecting functions is defined globally, then, the connecting functions must be linear regardless of linearity or otherwise of the vector field. As mentioned in Section 1, Spijker showed [129] that an ordinary differential equation (a scalar system) in the form of (5.3), subjected to some conditions, can
only accept linear connecting functions as expressed in (5.12). Inspired by Spijker’s theorem, we first prove a similar result for a system of second order autonomous differential equations (a vector force field).\(^9\)

**Theorem 5.1.** Let \( f \) in (5.3) satisfy the conditions

1. \( f \) is a real valued, well defined and continuous on the \( \mathbb{R}^n \).

2. \( \lim_{\zeta \to \infty} \zeta^{-2} M_i(q, \zeta) = 0 \) where \( M_i(q, \zeta) = \max\{ |f_i(q, \dot{q})| : |\dot{q}| \leq \zeta, i = 1, \ldots, N \} \).

and let \( F = F(x_1, \ldots, x_k) \) be a connecting function for the system (5.3). Then \( F \) is a linear connecting function.

**Proof.** See Appendix 6.

### 5.2.2 Forms of Nonlinear Connecting Functions: The Necessary Condition

In this section, we study a different form of connecting functions for the same system of second order autonomous (linear or nonlinear) differential equations. In this regard, instead of using the form suggested by Spijker, i.e.

\[
F = F(x_1, \ldots, x_k), \quad F, x_1, \ldots, x_k \in \mathbb{R}^n \tag{5.13}
\]

we propose a connecting function that is of the form

\[
\Phi = \Phi(x_1, \ldots, x_k, \dot{x}_1, \ldots, \dot{x}_k), \quad \Phi, x_1, \ldots, x_k, \dot{x}_1, \ldots, \dot{x}_k \in \mathbb{R}^n, \tag{5.14}
\]

allowing \( x_i \) and \( \dot{x}_i \) to be independent variables. We will show that, by revising the form of the connecting function in this way, it can now allow for nonlinear connecting functions. Moreover, we prove that the form of the connecting function depends on the order of the derivatives (velocity variables) in the vector sequence.

\(^9\)Spijker proved it for a scalar autonomous differential equation with an arbitrary order.
field $f$. As a result, for the system defined in Eq.(5.3), we show that the connecting functions are bi-directionally linear. In other words, each element of the connecting function $\Phi$ satisfies,

$$ J_{x_i}(\nabla_{x_j} \Phi_k) = J_{x_i}(\nabla_{\dot{x}_j} \Phi_k) = 0, \quad (5.15) $$

or, equivalently, accepts the general form

$$ \Phi(x_1, ..., x_m, \dot{x}_1, ..., \dot{x}_m) = d + \sum_{i=1}^{m} A_i x_i + \sum_{i=1}^{m} B_i \dot{x}_i + \sum_{k=1}^{n} \sum_{i=1}^{m} \sum_{j=1}^{m} [x_i^T C_{ijk} \dot{x}_j] e_k, \quad (5.16) $$

where $d \in \mathbb{R}^n$ and $A_i$, $B_i$ and $C_i$ are $n \times n$ real matrices. Moreover, we show that connecting functions of non-conservative systems (with first order derivative variables, i.e. with no $\dot{x}_{ij}^k$, $k \geq 2$ in $f$) satisfy

$$ f_j^T \left\{ J_{\dot{x}_j} (\nabla_{\dot{x}_k} \Phi) \right\}^T + (\nabla_{\dot{x}_k} \Phi_k)^T (J_{\dot{x}} f)_j = 0. \quad (5.17) $$

**Theorem 5.2.** Let $f$ in (5.3) satisfy the conditions

1. $f$ is a real valued, well defined and continuous on the $\mathbb{R}^n$.

2. $\lim_{\zeta \to \infty} \zeta^{-s} M_i(q, \zeta) = 0$, $s = 1, 2$ where $M_i(q, \zeta) = \max\{|f_i(q, \dot{q})|: |\dot{q}_i| \leq \zeta, i = 1, ..., n\}$,

and let $\Phi$ be a connecting function for the system (5.3). Then

$$ J_{x_i} (\nabla_{x_j} \Phi_k) = J_{x_i} (\nabla_{\dot{x}_j} \Phi_k) = 0, \quad s = 1 $$

$$ J_{x_i} (\nabla_{x_j} \Phi_k) = f_j^T \left\{ J_{\dot{x}_j} (\nabla_{\dot{x}_k} \Phi) \right\}^T + (\nabla_{\dot{x}_k} \Phi_k)^T (J_{\dot{x}} f)_j = 0, \quad s = 2. \quad (5.18) $$

**Proof.** See Appendix 7.

### 5.2.3 Existence of Nonlinear Connecting Functions: The Sufficient Condition

As mentioned before, although the result in the previous section provides a form for connecting functions of nonlinear systems, it does not guarantee that they exist, i.e. satisfy the governing equations of the
system. In other words, the aforementioned result only states that if a solution of a nonlinear system can be expressed as a function of a set of its other solutions, that function accepts the provided form. In this section we provide the sufficient condition for any connecting function to exist.

**Theorem 5.3.** Let $f$ in (5.3) satisfy the condition 1 in Theorem 5.1. Then the function $\Phi = \Phi(x_1, \ldots, x_k, \dot{x}_1, \ldots, \dot{x}_k)$, where $\{(x_k, \dot{x}_k) | k \in \mathbb{N}\}$ represents an arbitrary set of solutions of the system, is a connecting function for the vector field $f$, if and only if, there exist a real vector $d \in \mathbb{R}^n$ and real $n \times n$ matrices $A_i = [a_{i1} | \ldots | a_{in}]$, $B_i = [b_{i1} | \ldots | b_{in}]$, $C_{ijk}$, $i, j, k = 1, \ldots, n$ $D_{ij} = \sum_{k=1}^{n} C_{ijk}(T_1, \ldots T_m)$ and that satisfy the following system of functional equations

$$
\begin{align*}
\sum_{i=1}^{m} m \sum_{j=1}^{m} f^T(x_i) D_{ij} \dot{x}_j &+ \sum_{i=1}^{m} \left[ a_{ik} + \sum_{k=1}^{n} \sum_{j=1}^{m} C_{ijk} \dot{x}_j e_k \right]^T f(x_i) \\
+ \sum_{i=1}^{m} \sum_{j=1}^{m} \dot{x}_i^T D_{ij}^T f(x_j) &+ \sum_{i=1}^{m} \left[ b_{ik} + \sum_{k=1}^{n} \sum_{j=1}^{m} C_{ijk} x_i e_k \right]^T (J^T f(x_i) \dot{x}_i) = f_k(d + \sum_{i=1}^{m} A_i x_i + \sum_{i=1}^{m} B_i \dot{x}_i + \sum_{k=1}^{n} \sum_{i=1}^{m} \sum_{j=1}^{m} x_i^T C_{ijk} \dot{x}_j e_k) & k = 1, \ldots, n.
\end{align*}
$$

*Proof.* See Appendix 8.

### 5.3 Identification of Connecting Functions: Problem Statement

Note that, the problem of finding arbitrary solutions of nonlinear systems using connecting functions is not relevant anymore. That is because, unlike in linear systems, the existence of connecting functions for nonlinear systems can only be guaranteed locally. This is a direct implication of the sufficient condition provided in Theorem 5.3 which may not always be satisfied by any arbitrary set of the defining coefficients of nonlinear connecting functions, i.e. $d$, $A_i$, $b_i$ and $C_{ijk}$.

Therefore, in this section we present an approach to address the problem of finding a connecting function that generates a new fixed solution of the nonlinear system. Similar to finding a fixed solution of a linear system, this approach is meant to generate a balanced system of algebraic equations.
In this regard, we start by considering a connecting function as \( \Phi : (\bar{S}, \dot{\bar{S}}) \rightarrow S \), where \( \bar{S} \) represents the set of periodic solutions of the system (and \( \dot{\bar{S}} \) as its time derivative) and \( S \) represents the set of all solutions of the system. Next, we restrict the function \( \Phi \) to be a connecting function that (re)generates a solution of the system \( u(t) \) (and \( v(t) \) as its time derivative). In other words

\[
\Phi(\bar{x}_1(t), \ldots, \bar{x}_m(t), \dot{\bar{x}}_1(t), \ldots, \dot{\bar{x}}_m(t)) = u(t) \\
\frac{d\Phi}{dt}(\bar{x}_1(t), \ldots, \bar{x}_m(t), \dot{\bar{x}}_1(t), \ldots, \dot{\bar{x}}_m(t)) = v(t) \\
\frac{d^2\Phi}{dt^2}(\bar{x}_1(t), \ldots, \bar{x}_m(t), \dot{\bar{x}}_1(t), \ldots, \dot{\bar{x}}_m(t)) = f(u(t))
\] (5.20)

Note that the third set of equations in Eq. (5.20) is the same as the sufficient condition in Theorem 5.3. Then, in order to generate a set of algebraic equations from Eq. (5.20), we first substitute \( \Phi \) with its general form provided by Theorem 5.2, i.e.

\[
\Phi(\bar{x}_1(t), \ldots, \bar{x}_m(t), \dot{\bar{x}}_1(t), \ldots, \dot{\bar{x}}_m(t)) = d + \sum_{i=1}^{m} A_i x_i + \sum_{i=1}^{m} B_i \dot{x}_i + \sum_{k=1}^{n} \sum_{i=1}^{m} \sum_{j=1}^{m} [x_i^T C_{ijk} \dot{x}_j] e_k
\] (5.21)

in which we further substitute the periodic solutions \( \bar{x}_i(t) \) (and \( \dot{\bar{x}}(t) \)) by

\[
\bar{x}_i(t) = \gamma_i(T_i) + \sum_{j=1}^{N} \left[ \alpha_{ij}(T_i) \cos(j \frac{2\pi}{T_i} t) + \beta_{ij}(T_i) \sin(j \frac{2\pi}{T_i} t) \right] \\
\dot{\bar{x}}_i(t) = \frac{2\pi}{T_i} \sum_{j=1}^{N} j \left[ \beta_{ij}(T_i) \cos(j \frac{2\pi}{T_i} t) - \alpha_{ij}(T_i) \sin(j \frac{2\pi}{T_i} t) \right].
\] (5.22)

Note that we assume that all the parameters defining the periodic solutions, such as \( \gamma_i \), are already identified as functions of their periods, i.e. \( \gamma_i = \gamma_i(T_i) \). This way, \( \Phi \) can be represented as a parametric function expressed by

\[
\Phi = \Phi(d, A_{i=1}^{m}, B_{i=1}^{m}, C_{i,j=1,k=1}^{m,n}, \tau)
\] (5.23)

where \( \tau = [T_1 \ldots T_m] \) is the period vector associated with the periodic solution in \( \bar{S} \).
Second, $u(t)$ and $v(t)$ are replaced by their Taylor series expansion (of an appropriate order) such as

\[
\begin{align*}
    u(t) &= u + vt + \frac{1}{2} f(u)t^2 + \frac{1}{3!} f_x(u)vt^3 + \frac{1}{4!} \left[ f_x(u)f(u) + \sum_{k=1}^{n} \left( v^T H_x(u)v \right) e_k \right] + O(t^5) \\
    v(t) &= v + f(u)t + \frac{1}{2} f_x(u)vt^2 + \frac{1}{3!} \left[ f_x(u)f(u) + \sum_{k=1}^{n} \left( v^T H_x(u)v \right) e_k \right] t^3 + O(t^4)
\end{align*}
\]

where $u = u(0)$ and $v = v(0)$.

Finally, after the above substitutions, we evaluate the entire set of equations at $t_l = (l-1)t_\ast$, $l = 1, \ldots, n + (2m + m^2)n^2$, $t_\ast \ll 1$ to obtain

\[
\begin{align*}
    \Phi(d, A_{i=1}^m, B_{i=1}^m, C_{i,j=1,k=1}^{m,n}, \tau)|_{t=t_l} &= u(t_l) \\
    \frac{d}{dt} \Phi(d, A_{i=1}^m, B_{i=1}^m, C_{i,j=1,k=1}^{m,n}, \tau)|_{t=t_l} &= v(t_l) \\
    \frac{d^2}{dt^2} \Phi(d, A_{i=1}^m, B_{i=1}^m, C_{i,j=1,k=1}^{m,n}, \tau)|_{t=t_l} &= f(u(t_l)).
\end{align*}
\]

This process generates a set of nonlinear algebraic equations which can be represented in a much simpler form as

\[
g(\Phi(d, A_{i=1}^m, B_{i=1}^m, C_{i,j=1,k=1}^{m,n}, \tau), u, v) = 0.
\]

In other words, the problem of identifying the connecting function $\Phi$ of a set of known solution identified by the period vector $\tau$, reduces down to finding the vector $d$ and the matrices $A_{i=1}^m, B_{i=1}^m$ and $C_{i,j=1,k=1}^{m,n}$ such that for a given $(u, v)$ the above system of nonlinear algebraic equations is satisfied at all instances $t_l$.

Unlike linear algebraic systems which often have a unique or no solution, nonlinear algebraic system may have infinitely many solutions that must be searched for using numerical search methods. These search methods often require a rather accurate initial guess in order to converge to a solution. Therefore, in order to identify a connecting function, one needs a good approximation of the mentioned unknown
variables, i.e. $d$, $A_{i=1}^m$, $B_{i=1}^m$ and $C_{i,j=1,k=1}^{m,n}$, and $\tau$, in the above nonlinear algebraic system. The next section proposes three different approaches of providing such approximations.

### 5.4 Continuous Identification of Connecting Functions

In this section we propose three different numerical approaches in order to solve the system nonlinear algebraic equation defined in Eq.(5.26). Each proposed approach is then applied to the same two DOF nonlinear system to provide examples of its connection functions. In all examples provided in this study, in order to reduce the complexity of the algebraic system in Eq.(5.26), we use a simplified version of the connecting function by assuming that

$$
\begin{align*}
  d &= 0 \\
  A_i &= a_i I \\
  B_i &= b_i I \\
  C_{ijk} &= c_{ijk} I \\
  i, j, k &= \{1, 2\}
\end{align*}
$$

or equivalently

$$
\Phi = \Phi(\tau) = \sum_{i=1}^{2} a_i(\tau) I \dot{x}_i + \sum_{i=1}^{2} b_i(\tau) I \ddot{x}_i + \sum_{l=1}^{2} \sum_{i=1}^{2} \sum_{j=1}^{2} \dot{x}_i^T c_{ijl}(\tau) I \ddot{x}_j e_l.
$$

This assumption, although it was merely heuristic, proved adequate to identify an accurate connecting function for this system, and the number of free parameters reduced from 50 to 12 so the computational cost was greatly reduced.

#### 5.4.1 Connecting Functions as Solutions of a Homotopy Relation

Homotopy, in short, represents a relation that describes a continuous variation/deformation from one function to another. In other words, two functions are homotopic if one can be obtained by a continuous
deformation (homotopy) of the other. For example, consider the two functions $\sin(\pi x)$ and $2x^2(x - 1)$ in the interval $x \in [0, 1]$. One can construct a homotopy $\mathcal{H} : \sin(\pi x) \sim 2x^2(x - 1)$ as

$$\mathcal{H}(x, s) = (1 - s) \sin(\pi x) + s \left[2x^2(x - 1)\right]. \quad (5.28)$$

In this way, Fig. 5.5 shows that by changing the homotopy parameter, i.e. $s : 0 \to 1$, the function $\sin(\pi x)$ continuously deforms into the function $2x^2(x - 1)$.

Figure 5.5: A homotopy $\mathcal{H}(x, s)$ between $\sin(\pi x)$ and $2x^2(x - 1)$.s

The concept of homotopy can be also used in solving nonlinear algebraic equations such as the one presented in Eq. (5.26). The underlying idea in homotopy analysis is quite simple. Suppose that $\Phi$ is an unknown solution of a nonlinear algebraic system of equations

$$g(\Phi(\tau), u, v) = 0 \quad (5.29)$$

but $\Phi_0$, also a solution of the same system, is known. The unknown solution $\Phi$ represents an unknown connecting function defined by an unknown set of parameters $(d, A_i^{m=1}, B_i^{m=1}, C_i^{m,n}_{i,j=1,k=1}, \tau)$ for a given $(u, v)$ and similarly $\Phi_0$ represent a known connecting function defined by a known set of parameters $(d, A_i^{m=1}, B_i^{m=1}, C_i^{m,n}_{i,j=1,k=1}, \tau_0)$ obtained at $(u_0, v_0)$. 
This way, using the concept of Homotopy, we assume that there exist a continuous deformation \( \Phi(s) \) which starts as \( \Phi_0 \) at \( s = 0 \) and smoothly morphs into \( \Phi \) at \( s = 1 \) where \( s \) is the homotopy parameter. Such a homotopy, i.e. \( H : \Phi_0 \sim \Phi \), can be expressed by

\[
H(\bar{\Phi}, s) = g(\bar{\Phi}(\bar{\tau}(s)), \bar{u}(s), \bar{v}(s)).
\] (5.30)

wherein \((\bar{u}, \bar{v})\), which represent a homotopy path in the state space, and \(\bar{\tau}(s)\) have to be yet identified.

In order to simplify finding the homotopy path and the period vector, we present an example of how \(\bar{\tau}, \bar{u}, \bar{v}\) and \(s\) (and consequently their respective connecting functions) may be defined. In this regard, \(\bar{u}\) can be defined as any path with ascending potential energy \(E\) that starts from \(u_0\). In this way, for a conservative system, one potential definition can be provided by

\[
\begin{align*}
\bar{u} &= \{ \bar{u}(r) | \bar{u}(0) = u_0, \frac{d}{dr}E(\bar{u}) \neq 0 \} \\
\bar{v} &= 0 \\
s &= \frac{E(\bar{u}) - E(u_0)}{E(u) - E(u_0)} \in [0, 1], \ E(u_0) \to 0 \\
\bar{\tau} &= \{ \bar{\tau}(s) | \bar{\tau}(0) = \tau_0 \}. 
\end{align*}
\] (5.31)

The ascending-potential-energy condition is enforced so that the homotopy parameter \(s\), as defined in the above equation, changes monotonically from 0 to 1 which is a necessary condition for a homotopy parameter. Also, without loss of generality and in order to further simplify the homotopy problem, \(\bar{v}\) is set to be identical to zero. Moreover, the condition \(E(u_0) \to 0\) was enforced so that the period vector \(\bar{\tau}\) has a unique starting point. That is because, at \(u_0\) with very small potential energy, for example when the point \((u_0, v_0 = 0)\) is very close to the equilibrium, the period vector \(\bar{\tau}\) can only include the periods of linear periodic solutions of the system, i.e. \(\bar{\tau}(0) = \tau_0\). In this way, since \(E(u_0) \to 0\) implies that dynamics of the system is predominantly governed by the underlying linear system, the known connecting function, i.e. \(\Phi_0\), can be very well approximated at \((\tau_0, u_0, v_0 = 0)\) by a linear connecting.
function. In other words, $\Phi_0$ can be generalized as a function of linear periodic solutions of the system $\bar{x}_i, i = 1, \ldots, n$ as

$$\Phi_0 = \sum_{i=1}^{n} \bar{a}_i \bar{x}_i.$$  \hfill (5.32)

Therefore, by comparing the above equation and the general form of the connecting functions, i.e. $\Phi_0 = \Phi_0(d, A_{i=1}^{n}, B_{i=1}^{n}, C_{i,j=1,k=1}^{n,n}, \tau)$, one can readily write

$$A_i = \bar{a}_i I_n$$

$$\tau = \tau_0$$ \hfill (5.33)

$$d = 0, B_i = 0_{n \times n}, C_{ijk} = 0_{n \times n}$$

in which, $\tau_0$ include the periods of linear periodic solutions of the system and $\bar{a}_i$ are the solution of the linear algebraic system

$$[\bar{x}_1(0) \cdots \bar{x}_n(0)] \begin{bmatrix} \bar{a}_1 \\ \vdots \\ \bar{a}_n \end{bmatrix} = u_0.$$ \hfill (5.34)

To this end, all the unknowns in the Eq.5.30, i.e. $\Phi, \bar{\tau}, \bar{u}, \bar{v}$, are defined at $s = 0$. The next challenge is to find these unknowns as the homotopy parameter $s$ changes from 0 to 1. In order to define $\bar{u}$ we must know the contours of constant potential energy on the plane $\Omega$ to make sure $\bar{u}$ never becomes tangent to these contours. Moreover, the period vector $\bar{\tau}$ must be known as a function of the homotopy parameter. Figure 5.6 shows the potential energy function $E(u)$ and also how the inverse of period vector $\bar{\tau}$, i.e. the frequency vector, varies with energy for the two DOF system. As shown in this figure, finding a homotopy path $\bar{u}$ and then a period vector as a function of the homotopy parameter $s$ can be readily achieved.
Figure 5.6: Left: The path $\bar{u}$ in the phase space and on the (potential) energy surface. Right: As the homotopy path $\bar{u}$ changes from a low energy state $u_0$ to an arbitrary state $u$, the homotopy parameter $s$ changes from 0 to 1 monotonically. As a result, the period vector becomes known as functions of the homotopy parameter $s$.

For example, consider two lines passing through the origin and the points $u_0^{(1)} = [-0.006, 0.042]^T$ and $u_0^{(2)} = [-0.0045, 0.014]^T$. As it is evident in Fig. 5.8 (top-left) these two lines cross the potential energy contours transversely. Therefore, these two lines provides two examples of homotopy paths $\bar{u}_1$ and $\bar{u}_2$. Now that we have two paths that satisfy all the conditions above and the period vector $\bar{\tau}$ is known as a function of the potential energy of the system at every point along each of these path, we can move along paths $\bar{u}_1$ and $\bar{u}_2$ in small steps, and find new solutions, i.e. connecting functions. In this regard, by moving along a homotopy path, herien described by

$$\bar{u}_{k+1} = \bar{u}_k + \delta \bar{u}_k$$

$$\bar{u}_0 = u_0$$

we can update the values of the homotopy parameter $s$ and consequently $\bar{\tau}(s)$, at each step, as
\[ s_{k+1} = \frac{E(\bar{u}_{k+1}) - E(u_0)}{E(u) - E(u_0)} \]

\[ \bar{\tau}_{k+1}(s) = \bar{\tau}(s_{k+1}). \]  

Once, \( s_{k+1}, \bar{\tau}_{k+1}, \bar{u}_{k+1} \) (and \( \bar{v}_{k+1} = \bar{v}_k = 0 \)), are computed, one then can use them to predict the next connecting function, i.e. \( \bar{\Phi}_k(\bar{\tau}_{k+1}) \) as a function of the new set of periodic solutions with the period vector \( \tau_{k+1} \) and a new set of coefficients for \( (d, A_{i=1}^m, B_{i=1}^m, C_{i,j=1,k=1}^{mn}) \) obtained by

\[ d_i = d_i(s_k) + \nabla d_i(s_k)(\bar{\tau}_{k+1} - \bar{\tau}_k) \] (5.37)

\[ a_{ij}(s_{k+1}) = a_{ij}(s_k) + \nabla a_{ij}(s_k)(\bar{\tau}_{k+1} - \bar{\tau}_k) \] (5.38)

\[ b_{ij}(s_{k+1}) = b_{ij}(s_k) + \nabla b_{ij}(s_k)(\bar{\tau}_{k+1} - \bar{\tau}_k) \] (5.39)

\[ c_{ijl}(s_{k+1}) = c_{ijl}(s_k) + \nabla c_{ijl}(s_k)(\bar{\tau}_{k+1} - \bar{\tau}_k) \] (5.40)

and use it as an initial guess to solve the algebraic system in Eq. (5.26), i.e.

\[ \text{Solve } g(\bar{\Phi}(\bar{\tau}), \bar{u}_{k+1}\bar{v}_{k+1}) = 0 \]  

with initial guess \( \bar{\Phi}^* = \bar{\Phi}_{k+1} \).  

Note that the gradient vectors \( \nabla a_i, \nabla b_i \) and \( \nabla c_{ijl} \) must be approximated numerically at each step. This process, as described in Fig. 5.7, repeats until one reaches the point \( u \) on the homotopy path.
The homotopy problem can be solved continuously to find new connecting functions. If the initial guess for the next connecting function, i.e. the predicted connecting function, does not lead to a connecting function, one has to repeat the prediction step with a smaller change in the homotopy parameter $\delta s$.

Figure 5.8 shows the two mentioned examples of the homotopy paths, as well as, two examples of connecting functions associated with two points on the homotopy paths. Since both examples of the connecting functions are associated with points with the same potential energies $E(\bar{u}_k^{(1)}) = E(\bar{u}_k^{(2)})$ and consequently the same homotopy parameters $s_k^{(1)} = s_k^{(2)}$, they both use the same set of two periodic solutions of the two DOF system.

### 5.4.2 Identification of Connecting Functions Using Continuation

The problem of defining homotopy paths, a homotopy parameter and then the period vector as a function of the homotopy parameter is not always easy to solve and may demand a tremendous initial knowledge for larger, more complex systems. One way to avoid this problem is to use a continuation technique. Continuation techniques, similar to homotopy analysis methods, require a set of known solutions as an initial guess, which can then be evolved into new solutions as the homotopy parameter is varied.
initial state to find new solutions on a continuous path, however this time, the continuation path $\vec{u}$ need not to be defined but solved for in an iterative process.

In this process, assuming that the initial state $(u_0, v_0 = 0)$ identifies a known solution $\Phi_0$ as a linear connecting function, one uses the Newton’s method to predict how the the continuation path $\vec{u}$ and the period vector $\vec{\tau}$ may change by solving

$$
\begin{bmatrix}
\partial g_k \bigg| \Phi_k, \vec{\tau}_k, \vec{u}_k \vec{v}_k \\
\partial g_k \bigg| \Phi_k, \vec{\tau}_k, \vec{u}_k \vec{v}_k
\end{bmatrix}
\begin{bmatrix}
\Delta \vec{u}_k \\
\Delta \vec{\tau}_k
\end{bmatrix}
= 0
$$

(5.42)

wherein

$$
\begin{align*}
\partial g_k &= \partial g_k \bigg| \Phi_k, \vec{\tau}_k, \vec{u}_k \vec{v}_k \\
\partial g_k &= \partial g_k \bigg| \Phi_k, \vec{\tau}_k, \vec{u}_k \vec{v}_k
\end{align*}
$$

(5.43)

to find the next points on the continuation path by

$$
\begin{align*}
\vec{u}_{k+1} &= \vec{u}_k + r \Delta \vec{u}_k \\
\vec{\tau}_{k+1} &= \vec{\tau}_k + r \Delta \vec{\tau}_k.
\end{align*}
$$

(5.44)

in which $r$ is the step-size. Note that we still assume that $\vec{v}_{k+1} = \vec{v}_k = 0$.

Once, $\vec{\tau}_{k+1}$, $\vec{u}_{k+1}$, are computed, one can use them to update the algebraic system define by $g$, which then can be solved for a new connecting function by using $\Phi_{k+1}(\tau_{k+1})$ as the initial guess, i.e.

$$
\text{Solve } g(\Phi_{k+1}, \vec{\tau}_{k+1}, \vec{u}_{k+1} \vec{v}_{k+1}) = 0
$$

with initial guess $\Phi^* = \Phi_{k+1}$.

(5.45)

Similar to the homotopy analysis method, the initial guess for the connecting function $\Phi_{k+1}(\tau_{k+1})$ is obtained by updating it’s defining set of coefficients for $(d, A_{i=1}^m, B_{i=1}^m, C_{i,j=1}^{m,n})$ as
Figure 5.8: Top-Left: Two examples of homotopy paths for the two DOF system on $\Omega$ plane. Top-Right: A set of two periodic solutions of the two DOF system used in the connecting functions associated with two points on the homotopy paths. Bottom: Comparison between predictions made by two connecting functions of solutions of the two DOF system and the accurate solutions of the system obtained using numerical integration.
\[ d_i = d_i(s_k) + r \nabla_{\Phi} d_i(s_k) \Delta \tau_k \]  
\[ a_{ij}(s_{k+1}) = a_{ij}(s_k) + r \nabla_{\Phi} a_{ij}(s_k) \Delta \tau_k \]  
\[ b_{ij}(s_{k+1}) = b_{ij}(s_k) + r \nabla_{\Phi} b_{ij}(s_k) \Delta \tau_k \]  
\[ c_{ijkl}(s_{k+1}) = c_{ijkl}(s_k) + r \nabla_{\Phi} c_{ijkl}(s_k) \Delta \tau_k. \]  

Note that, if the initial guess \( \Phi_{k+1} \) does not lead to a new connecting functions, one must reduce the step-size \( r \) and recalculate the next point on the continuation path, i.e. \( \tau_{k+1}, \Phi_{k+1} \). A more general version of this process is depicted in Fig. 5.9.

---

**Figure 5.9:** The nonlinear algebraic system can be solved continuously to find new connecting functions. If the initial guess for the next connecting function, i.e. the predicted connecting function, does not lead to a connecting function, one has to repeat the prediction step with a smaller change in the homotopy parameter \( \delta s \).

This continuation technique was used for the same initial points as the one used in the homotopy analysis methods, i.e. \( \mathbf{u}_0^{(1)} = [-0.006, 0.042]^T \) and \( \mathbf{u}_0^{(2)} = [-0.0045, 0.014]^T \), to generate two continuation paths as shown in Fig. 5.10. It is worth noting that, unlike the homotopy paths in homotopy analysis method, solutions on the continuation paths and periodic solution from the two main branches may not (and in this case do not) share the same potential energies. That is because, in the continuation
technique, neither the point on the continuation paths nor the periodic solutions used in the connecting function associated with that point are constrained to have the same potential energy.

Finally, Fig. 5.11 shows all the continuation and homotopy paths obtained for the two DOF system. It is worth pointing out that, unlike the homotopy paths which can be defined so that they pass through a point where one desires to compute the response of the system (and hence the connecting function), the continuation paths are determined by the dynamics of the algebraic system defined by $g$ so that the change required to obtain the next connecting function is minimized. Therefore, the continuation paths, unlike homotopy paths, cannot be navigated to pass through an arbitrary point.

5.5 Point-wise Identification of Connecting Functions

The previous section provided two numerical approaches to identify connecting functions on continuous paths of solutions. However, in order to identify an unknown nonlinear connecting function one always had to start from a known linear one very close to the equilibrium of the nonlinear system. In this section, we provide another approach that, in essence, will replace the known linear connecting function very close to the equilibrium of the system with a linear approximation of the same nonlinear connecting function. Let’s explain this by a comparison shown in Fig. 5.12. In this figure, the accurate solution of the two DOF system is compared with predictions provided by a) a nonlinear connecting function $\Phi = \Phi(d, A_{i=1}^n, B_{i=1}^n, C_{i,j=1,k=1}^{n,n}, \tau)$ and b) the linear terms of the same nonlinear connecting function, i.e. $\Phi = \Phi(d, A_{i=1}^n, B_{i=1}^n, C_{i,j=1,k=1}^{n,n} = 0_{n \times n}, \tau)$. As it’s evident in these comparison, the linear part of the nonlinear connecting function may provide an acceptable initial guess for the nonlinear connecting function if not accurate predictions.
Figure 5.10: Top-Left: Two examples of continuation paths for the two DOF system on $\Omega$ plane. Top-Right: A set of two periodic solutions of the two DOF system used in the connecting functions associated with two marked points on the continuation paths. Bottom: Comparison between predictions made by two connecting functions of the two DOF system and the accurate solutions of the system obtained using numerical integration.
Figure 5.11: The continuation paths are tangent to the homotopy paths at the initial points $u_1^{(1)}$ and $u_1^{(2)}$ yet they trace different paths in the state space.

Figure 5.12: A solution of the two DOF system obtained by numerical integration is compared with predictions provided by a nonlinear and it's underlying linear connecting function.

This observation inspired the development of a new approach that can identify an approximate linear connecting function at any arbitrary point in the state space regardless of its proximity to the equilibrium of the system. Such an approximation will provide the initial guess for solving the nonlinear algebraic system defined by $g$. 
In this approach, we first define a concept, dubbed almost orthogonality (of periodic solutions), and then use it to identify the aforementioned approximate linear connecting function. The definition of almost orthogonality and also the order of almost orthogonality follow next.

### 5.5.1 Almost Orthogonal Periodic Solutions

**Definition** - (Almost Orthogonality) Suppose \( S \) is an orthogonal basis for the Banach space \( \mathbb{B} \) and \( \bar{S}_k \) is an ordered subset of \( S \) defined as \( \bar{S}_k = \{ e_i | e_i \in S, i = 1, \ldots, k \} \). Then, any two arbitrary functions \( u, v \in \mathbb{B} \) are almost orthogonal if there exist at least one set \( \bar{S}_l, l \in \mathbb{N} \) such that the respective projections of \( u \) and \( v \) onto this set are orthogonal.

**Definition** - (Order of Almost Orthogonality) Suppose \( u, v \in \mathbb{B} \) are almost orthogonal with respect to the sets \( \bar{S}_i, i \in I \) where \( I \subset \mathbb{N} \) is an index set. Then the order of orthogonality of \( u \) and \( v \) is defined as

\[
O^\perp(u, v) = \max_{i \in I} |\bar{S}_i|.
\]  

(5.50)

Next, we state a theorem that provides a necessary and sufficient condition for a set of of period solutions to be mutually almost orthogonal.

**Theorem 5.4** - (Almost Orthogonality of Periodic Functions) Suppose \( R = \{ x_1, \ldots, x_m \} \) is a set of periodic functions in Banach space with their respective periods collected into the vector \( \tau = [T_1, \ldots, T_m]^T, T_i \in \mathbb{R}^+ \). Then, \( x_i \in R, i = 1, \ldots, m \) are mutually almost orthogonal of order \( K \), if and only if the period vector \( \tau \) is commensurable, i.e. \( \tau = Tp, T \in \mathbb{R}^+, p_i \in \mathbb{N}, \gcd(p_i, p_j)_{i \neq j} = 1 \), with at most one integer \( p_l \leq K, l \in \{ 1, \ldots, m \} \).

**Proof.** See Appendix 9.

Let’s explain this concept with an example. Suppose \( x_1(t) \) and \( x_2(t) \) are two periodic functions and also suppose that \( \frac{T_1}{T_2} = \frac{5}{7} \). In this way, one can readily express the period vector for these two solution
This way, Theorem 5.4 claims that these two periodic functions are almost orthogonal of order 4. That is because, the number 4 is the largest natural number that satisfies \( 4 < \begin{bmatrix} 7 \\ 5 \end{bmatrix} \). Equivalently, this implies that the projections of these two functions on all subsets of an orthogonal basis, with up to their first 4 members, are orthogonal. Let's examine this claim. Since these functions are periodic, they both have a convergent Fourier series representations on the following orthogonal basis

\[
E = \left\{ \frac{1}{\sqrt{2\pi}} e^{ikt} | k \in \mathbb{Z}, t \in [0, 2\pi) \right\}
\]

as

\[
x_1(t) = \sum_{m \in \mathbb{N}} \left\{ A_{1m} \cos(m\omega_1 t) + B_{1m} \sin(m\omega_1 t) \right\}
\]

\[
x_2(t) = \sum_{m \in \mathbb{N}} \left\{ A_{2m} \cos(m\omega_2 t) + B_{2m} \sin(m\omega_2 t) \right\}.
\]

Therefore, the claim asserts that all partial sum Fourier series

\[
x_{1k}(t) = \sum_{m=1}^{k \leq 4} \left\{ A_{1m} \cos(m\omega_1 t) + B_{1m} \sin(m\omega_1 t) \right\}
\]

\[
x_{2k}(t) = \sum_{m=1}^{k \leq 4} \left\{ A_{2m} \cos(m\omega_2 t) + B_{2m} \sin(m\omega_2 t) \right\}
\]

are mutually orthogonal, i.e.

\[
\int_0^{2\pi} x_{1k} x_{2k} d\phi = 0, \phi = \omega_1 t/5 = \omega_2 t/7, k = 1, ..., 4,
\]

which can be easily verified knowing that all following identities hold.
\[
I_1 = \int_0^{2\pi} \cos(m_1 \phi) \cos(m_2 \phi) d\phi = 0 \\
I_2 = \int_0^{2\pi} \cos(m_1 \phi) \sin(m_2 \phi) d\phi = 0 \\
I_3 = \int_0^{2\pi} \sin(m_1 \phi) \sin(m_2 \phi) d\phi = 0
\]

(5.55)

, \ m_1, m_2 \leq 4 < \begin{bmatrix} 7 \\ 5 \end{bmatrix}.

Theorem 5.4 can also play a crucial role in the case where one seeks to search in a set of periodic solutions for a subset of mutually almost orthogonal solutions of a particular order \( K \). In this regard, let’s suppose that the system in (5.3) has \( n \) distinct periodic solutions and consequently at least \( n \) continuous branches of periodic solutions as depicted schematically in Fig. 5.13. This way, suppose we picked a periodic solution from the first branch with the frequency of \( f_1^* \) and wish to generate a set of \( n \) mutually almost orthogonal periodic solutions of an order \( K = 4 \). Then, according to theorem 5.4, one needs to first find \( n - 1 \) rational numbers \( \frac{p_2}{p_1}, \ldots, \frac{p_n}{p_1} \) such that \( p_i > 4, \ i = 1, \ldots, n \). Then, as depicted in Fig. 5.13, once the frequency lines defined by \( f_i = f_1^* \frac{p_i}{p_1} \) cross the branches of periodic solutions, the periodic solutions on these cross-sections will form the sought subset of mutually almost orthogonal solutions.

Figure 5.13: Process of generating a set of almost orthogonal periodic solutions. Selected periodic solutions are mutually almost orthogonal of order \( \min_i(p_i) - 1 \).
It is also worth mentioning that finding such rational numbers is not a hard task. That is because, the set of rational numbers such as \( \frac{p_i}{p_1} \), in any interval of real numbers such as \( (f_k^* - \epsilon, f_k^* + \epsilon) \) is dense. In other words, if one picks an arbitrarily natural number \( K > 1 \) as an almost orthogonality order, there exists at least one interval of frequencies on any branches of periodic solutions, i.e., the \( (f_k^* - \epsilon, f_k^* + \epsilon) \), for which there are infinitely many rational numbers \( \frac{p_i}{p_1} \) such that \( \frac{p_i}{p_1} f_k^* \in (f_k^* - \epsilon, f_k^* + \epsilon) \). This implies that for any order of almost orthogonality \( K \), one can always generate a denumerable (infinite but countable) set of almost orthogonal periodic solutions.

Figure 5.14 illustrates this fact for the two DOF nonlinear system. The left plot in this figure shows two continuous branches of periodic solutions starting from linear periodic solutions of its underlying linear system. One periodic solution from the (nonlinear region of) first branch was selected. Then all periodic solutions that are almost orthogonal to this solution with an order \( K = 15 \), by finding closest rational number \( \frac{p_i}{p_1} \) to \( \frac{\omega_i}{\omega_1} \) \(^{10}\) and using the criteria \( p_i, p_1 > 15 \), were identified. The plot on the right clarifies that the seemingly continuous interval of almost orthogonal periodic solutions on the second branch, in fact, consists of many small intervals. Moreover, it can be observed that, for this particular system, these intervals do not include the periodic solution with same energy as the one of the initial solution from the first branch. This is particularly interesting, because it differentiates the current identification approach from the homotopy analysis method discussed before as it does not impose that the known periodic solutions in the definition of connecting functions must have the same potential energies.

\(^{10}\)Using MATLAB’s rat function.
To this end, we complete this section with stating the following theorem without any proof.

**Theorem 5.5** - For any order of almost orthogonality, any arbitrarily small interval of any continuous branch of periodic solutions contains a denumerable set of almost orthogonal periodic solutions.

### 5.5.2 Pointwise Identification of Approximate Linear Connecting Function

In this section we present an approach that essentially breaks down the problem of solving the nonlinear algebraic system in Eq. (5.26) into two simpler problems. First, solve a linear version of the nonlinear algebraic system to find an approximate linear connecting function. Second, use the approximate linear connecting function as an initial guess to solve the original nonlinear system for the nonlinear connecting function. In this regard, recall that the nonlinear algebraic system was obtained from Eq. (5.20) which is repeated below for convenience.
\[
\Phi(d, A_{i=1}^m, B_{i=1}^m, C_{i,j=1,k=1}^{m,n}, \tau) = u(t)
\]
\[
\frac{d\Phi}{dt}(d, A_{i=1}^m, B_{i=1}^m, C_{i,j=1,k=1}^{m,n}, \tau) = v(t)
\]
\[
\frac{d^2\Phi}{dt^2}(d, A_{i=1}^m, B_{i=1}^m, C_{i,j=1,k=1}^{m,n}, \tau) = f(u(t)).
\]

The first step of this approach consists of two substitutions in the above nonlinear system, i.e. \( \Phi \) with \( \Phi_L = \Phi(d, A_{i=1}^m, B_{i=1}^m, C_{i,j=1,k=1}^{m,n}, \tau) = 0 \) and \( f \) with \( f_L(x) = J_f(u,v)x \), in order to generate a linear system of equations

\[
\Phi_L(d, A_{i=1}^m, B_{i=1}^m, \tau) = u(t)
\]
\[
\frac{d\Phi_L}{dt}(d, A_{i=1}^m, B_{i=1}^m, \tau) = v(t)
\]
\[
\frac{d^2\Phi_L}{dt^2}(d, A_{i=1}^m, B_{i=1}^m, \tau) = f_L(u(t)).
\]

In the second step, the period vector \( \tau \) is replaced by a period vector of a set of almost orthogonal periodic solutions, i.e. \( \tau^{(AO)} \). This substitution, allows one to perform a series of inner products of third set of equations, i.e. \( \frac{d^2\Phi_L}{dt^2}(d, A_{i=1}^m, B_{i=1}^m, \tau) = f_L(u(t)) \), and the almost orthogonal periodic solutions denoted by \( \bar{x}_i \) to generate a balanced set of linear algebraic equations. In other words,

\[
\Phi_L(d, A_{i=1}^m, B_{i=1}^m, \tau)|_{t=0} = u
\]
\[
\frac{d\Phi_L}{dt}(d, A_{i=1}^m, B_{i=1}^m, \tau)|_{t=0} = v
\]
\[
\int_0^{2\pi} \left[ \frac{d^2\Phi_L}{dt^2} - f_{Lk}(\Phi_L) \right] \bar{x}_{ij}(\phi)d\phi = 0
\]
\[
\int_0^{2\pi} \left[ \frac{d^2\Phi_L}{dt^2} - f_{Lk}(\Phi_L) \right] \hat{x}_{ij}(\phi)d\phi = 0
\]
\[
i = 1, \ldots, m, \ j, k = 1, \ldots, n,
\]

Note that, since the above set of algebraic equations can form an algebraic system with up to \( 2mn^2 + 2n \) equations which is more than the maximum number of unknowns \( 2mn^2 + n \), it can always have a solution.

Moreover, what makes this approach interesting is that one does not need to numerically integrate the system to find these equations and this is more than evident in the first part of the process that generates
the above set of equations, i.e. assuming \( \bar{x}_{ij} = \sum_{m=1}^{K-1} \left[ \alpha_{ijm} \cos(mp_{ij} \omega t) + \beta_{ijm} \sin(mp_{ij} \omega t) \right] \), one can write

\[
\int_0^{2\pi} \frac{d^2 \Phi_{kL}}{dt^2} \bar{x}_{ij}(\phi) d\phi = A_{ij} \|x_{ij}\|^2 - \pi B_{ij} (p_{ij} \omega)^2 \sum_{m=1}^{K-1} m^2 \alpha_{ijm}^2 \beta_{ijm}^2 \\
\int_0^{2\pi} \frac{d^2 \Phi_{kL}}{dt^2} \dot{x}_{ij}(\phi) d\phi = B_{ij} \|\dot{x}_{ij}\|^2 + \pi A_{ij} (p_{ij} \omega)^3 \sum_{m=1}^{K-1} m^3 \left( \alpha_{ijm}^2 - \beta_{ijm}^2 \right).
\]

(5.58)

In order to illustrate the performance of this approach, we apply it to a point on one of the two homotopy paths from the previous sections. In this example, the first periodic solution was picked using the \( \bar{\tau}(s) \) to have the same potential energy as the point \( u \). However the second periodic solution was replaced by one that is almost orthogonal to the first periodic solution in order to generate a set of almost orthogonal periodic solutions of order \( K = 15 \). Then the linear algebraic system in (5.57) was solved for four coefficients \( a_1, a_2, b_1 \) and \( b_2 \) to obtain the approximate linear connecting function. Next, the mentioned coefficients were used along with zero nonlinear coefficients, i.e. \( c_{ijk} = 0 \), \( i, j, k = \{1, 2\} \) as the initial guess for the nonlinear connecting function in solving the nonlinear algebraic system in Eq. (5.26). Figure 5.15 compares the results obtained by applying this approach.

Note that imposing the almost orthogonality condition on the periodic solutions, causes one to choose periodic solutions that may not have the same potential energies. Moreover, although the approximate linear connecting function cannot exactly match the prediction provided by the linear terms of the final nonlinear connecting function, they are close enough to provide an excellent initial guess for the nonlinear connecting function. These results show substantial progress towards a practical means of using nonlinear connecting functions to find the response of a nonlinear system. Further research is needed to understand the limitations of this approach and the computational savings that can be realized.
Figure 5.15: Comparison between predictions made by three connecting functions, i.e. a nonlinear, the linear terms of the nonlinear and an approximate linear, of the two DOF system and the accurate solutions of the system obtained using numerical integration.
Chapter 6

Conclusions and Suggestions for Future Work

In this study, we proposed a definition for a nonlinear mode of vibration as an eigensolution on a set of invariant manifolds of a nonlinear system, specifically, instantaneous center manifolds or ICMs. The proposed definition not only encompasses all the previous nonlinear mode definitions, known as NNM, but also leads to new methods of calculation of nonlinear modes that do not require any previously known solution as an initial guess. The significance of such methods becomes more evident when one consider two (unique) facts about nonlinear systems. First, the infinitely many nonlinear modes of a nonlinear system can not be traced back to the system’s linear modes in a continuous fashion. In this regard, we presented an extremely simple yet effective method, i.e. MMC, for finding periodic solutions of conservative nonlinear systems. The results, obtained thus far, show that the method consistently provides good accuracy. The method uses a similar condition to the one used in multiple-point shooting methods, however, it does not require integration of the vector field over any period of time and is capable of finding more than one periodic solutions of the nonlinear system in each solution. Although not presented in this document, the algorithm has been implemented in a continuation framework and
seems to be more computationally efficient than shooting algorithms. Second, those nonlinear modes (that are not continuations of linear modes) seem to be the reason for potentially infinitely many connecting functions to exist and at the same time the key to find them. The second property is the main motivation for stability and bifurcation analysis of nonlinear modes.

As explained in Chapter 4, although the methods of stability and bifurcation analysis are well established, it is incredibly difficult to perform these analysis accurately for a nonlinear system because the exponents approach zero and therefore numerical errors contaminate the prediction. In this regard, in Chapter 4, we presented an upper limit for two sources of error in the process of calculating Floquet exponents. The mentioned limits were used to propose a criteria for validation of near-zero Floquet exponents. Moreover, an algorithm was also proposed which adjusted the integration tolerance and approximation level for the periodic solutions using the criteria, to assure that valid Floquet exponents were calculated along an entire branch. The proposed algorithm also reduces the the computational cost of finding periodic solutions by providing an adaptive tolerance for the periodic orbit solver. The results presented herein, for a simple system, show that the criteria can successfully be used to generate valid Floquet exponents which can be consequently used in credible stability and bifurcation analysis.

In order to find the connecting functions of a nonlinear system, one must know the general form which these functions can accept. The main theorems presented in Chapter 5 by the author addresses this question. Moreover, the proposed numerical approaches show how one can uncover individual or branches of connecting functions. The first two approaches use homotopy analysis and continuation respectively to identify continuous branches of connecting functions. Because these connecting functions are local in nature, this amounts to computing a family of solutions to the nonlinear system that can be constructed from combinations of a small number of its nonlinear modes. However both methods demand starting from a known connecting function, i.e. a linear solution, at a point very close to the equilibrium point of the nonlinear system. The third method, however, overcomes this constraint by
finding a linear approximation of a connecting function at any arbitrary point in the state space and using it as the starting point in the search for the nonlinear connecting function. The concept of almost orthogonality and order of almost orthogonality is defined for periodic solutions. This concept makes it possible for the third approach to efficiently find the a linear approximation of connecting functions at any arbitrary point regardless of its proximity to the equilibrium of the system. In addition, a numerical approach is provided to select from among the periodic solutions of the system a small number of almost orthogonal periodic solutions (of any order) to use as a basis.

Two fundamental questions remain unanswered. First, how can one identify the connecting functions of a nonlinear system, knowing its eigensolutions and their stability status, over a confined region of the state space? Second, how can one reconstruct differential EOM a nonlinear system assuming that its connecting functions and eigensolutions are all known? The answer to this question is the key in developing both order reduction and sub-structuring.

6.1 Suggestions for Future Works

In this section we briefly outline some suggestions for follow-on research that can benefit greatly from this study.

- Identification of connecting functions over an entire confined region of the state space of the system.
- Use the connecting functions of a nonlinear system in order to construct reduced order models.

The theoretical foundation of the first can be summarized as follows. Assuming that $F$ is a connecting function, then any nonlinear mode of the system $q$ must be an eigensolution of $F$ because it satisfies

$$F(q) = q.$$  \hspace{1cm} (6.1)
Next, suppose that $G$ is the inverse of the connecting function $F$, i.e.

$$G = F^{-1}.$$ 

(6.2)

Therefore, since $G$ projects any solution of the system on a set of its eigensolutions, any nonlinear mode of the system $q$ is also an eigensolution of $G$. Thus, $F$ and $G$ accept the same set of eigensolution as the eigensolution of the system.

Now suppose that $g_A(s)$, $g_B(s)$ and $g_C(s)$ are footprints\textsuperscript{1} of $G$ on invariant manifolds $A$, $B$ and $C$ respectively which are tangent to each other at the eigensolution $q(s_0)$ on the invariant manifold $A$.

This situation (except for the invariant manifold $A$) is depicted in Fig. 6.1. Since the invariant manifolds are tangent, eigensolutions of the system will bifurcate at the solution $q(s_0)$ to continue on the surfaces of now three invariant manifolds. Two of these solutions are represented as $q_A(s_0 + \delta s)$ and $q_B(s_0 + \delta s)$. Since by the definition of a manifold the functions $g_A(s)$ and $g_B(s)$ can not be identical (or even cross each other), one can conclude that the footprint of the function $G$ bifurcates at $q(s_0)$ as well. Assuming that $F$ is surjective (as we did in Chapter 5) then $F$ will also bifurcate whenever $G$ does. Therefore one can conclude that the connecting functions of a nonlinear system bifurcate whenever and wherever its eigensolutions bifurcate. In other words, nonlinear systems accept as many connecting functions as they exhibit bifurcations.

The last conclusion (has been proven rigorously, although the proof is not included in this document) shall direct all the forthcoming efforts to calculate connecting functions of a nonlinear system by forcing the general form, given in Chapter 5, to bifurcate exactly where a nonlinear mode of the system bifurcates. The tangent vectors of the bifurcated footprints, i.e. $g_B(s)$ and $g_C(s)$, can be determined by finding eigenvectors of the tangent plane on either invariant manifold at $q(s_0)$. The later is the first step in the procedure to evaluate orientability that was discussed in [7], and was used to identify jump

\textsuperscript{1}This means that the domain of the function $G$ is limited to the points on the surfaces of each invariant manifold.
phenomena. Once a connecting function is identified around a bifurcation point, one can use the set of nonlinear modes in that vicinity to construct a reduced order model based on the identified connecting function.

Figure 6.1: Three invariant manifolds $A$ (not shown), $B$, and $C$ are tangent at the eigensolution $q(s_0)$ on the invariant manifold $A$ with bifurcation. Footprint of $G$, and consequently $F$, on invariant manifold $A$ bifurcates at the same eigensolution which results in two new footprints $g_B(s)$ and $g_C(s)$. 
Chapter 7

Appendices

7.1 Appendix 1: Almost Periodicity

Almost Automorphism - (Definition) Let $X$ be a real or complex Banach space and $f : \mathbb{R} \to X$ a strongly continuous function. Then $f$ is almost automorphic if for every sequence of real numbers $(s'_{n})$, there exists a subsequence $(s_{n})$ such that the function $g(t)$:

$$g(t) = \lim_{n \to \infty} f(t + s_{n})$$

is well defined for each $t \in \mathbb{R}$ and

$$\lim_{n \to \infty} g(t - s_{n}) = f(t)$$

for each $t \in \mathbb{R}$ [100].

Almost Periodicity - (Definition) If the convergence in definition of almost automorphism is uniform on $\mathbb{R}$, then $f$ is almost periodic [27]. Almost automorphism is thus more general than almost periodicity.

Almost Automorphism - (Alternative Definition) A continuous function $f : \mathbb{R} \times X \to X$ is said to be almost automorphic in $t \in \mathbb{R}$ for each $x \in X$ if for every sequence of real numbers $(s'_{n})$ there exists
a subsequence \((s_n)\) such that the function \(g(t, x)\):

\[
g(t, x) = \lim_{n \to \infty} f(t + s_n, x)
\]

is well defined for each \(t \in \mathbb{R}\) and each \(x \in X\) and

\[
\lim_{n \to \infty} g(t - s_n, x) = f(t, x)
\]

exists for each \(t \in \mathbb{R}\) and each \(x \in X\) [100].

An almost periodic function is a function whose value is “approximately” repeated when its argument is increased by “properly selected constants”, i.e. its almost periods. Moreover, a continuous function \(f(x) : \mathbb{R} \to \mathbb{R}\) is almost periodic if for every \(\epsilon > 0\) it is possible to find a quantity \(l > 0\) such that every interval of the \(x\)-axis of length \(l\) contains at least one almost period of the function \(f(x)\), i.e. there is at least one number \(0 < \tau \leq l\) (almost period) such that

\[
|f(x + \tau) - f(x)| < \epsilon, \forall x \in \mathbb{R}.
\]

This definition is different from the one provided for the periodic functions by the fact that it allows for approximate retraction instead of exact repetition and accepts possibly many numbers as almost periods.

Simple examples of almost periodic functions that are not periodic can be obtained by adding trigonometric periodic functions with incommensurable periods, such as \(\cos(2x) + \cos(2\sqrt{3}x)\). [98] has shown these sums are not periodic and one can readily show that they satisfy the Riesz-Fisher Theorem which makes them almost periodic. For the mentioned example, one can also confirm that for \(\epsilon = \{0.6, 0.06, 0.006\}\), almost periods respectively are \(\tau = \{15.4415, 40.8111, 87.9692\}\). An analytical example of invariant manifolds for almost periodic orbits is provided in [10]. There are several other equivalent definitions provided for almost periodic functions [36, 97, 112].
7.2 Appendix 2: Hilbert Transform

**Hilbert Transform** (Definition) - When this integral exists, the Hilbert transform of a continuous function $f(t)$ is defined as

$$
H_f(t) = \frac{1}{\pi} PV \int_{-\infty}^{+\infty} \frac{f(\tau)}{t-\tau} d\tau = \frac{1}{\pi} \lim_{R \to \infty} \int_{-R}^{+R} \frac{f(\tau)}{t-\tau} d\tau,
$$

where the Cauchy principal value of a doubly infinite integral of a function $f(t)$ is defined by

$$
PV \int_{-\infty}^{+\infty} f(t) dt = \lim_{R \to \infty} \int_{-R}^{+R} f(t) dt.
$$

[66, 112].

The Hilbert transform does not exist for all functions, as the improper integral (7.1) must be convergent. However, the Hilbert transform is well-defined for a broad class of functions, i.e. all measurable functions $f$ with finite p-norms or equivalently $\|f\|_p := (\int |f|^p dt)^{1/p} < \infty$, namely those in $L^p(R)$ for $1 < p < \infty$ [112, 97].

**Strong Analytic Signal** - A strong analytic signal is the complex continuous time-domain signal $f(t)$ having the following property:

$$
H_f(t) = -if(t)
$$

If the strong analytic signal $f(t)$ is separated into its real and imaginary parts, i.e. $f(t) = g(t) + ih(t)$, then

$$
H_f(t) = H_g(t) + iH_h(t) = -i[g(t) + ih(t)]
$$

$$
\Rightarrow H_g(t) = h(t), \; H_h(t) = -g(t)
$$

In other words,
\[ H_{Re[f]}(t) = \text{Im}[f(t)] \]

As a consequence, any signal \( S(t) \) in the form of

\[ S(t) = x(t) + iH_x(t) \]

is a strong analytic signal.

The Hilbert transform can be used to create an analytic signal from a real signal. Instead of studying the signal in the frequency domain it is possible to look at a rotating vector with an instantaneous phase \( \phi(t) \) and an instantaneous amplitude \( a(t) \) in the time domain, that is

\[ S(t) = x(t) + iH_x(t) = a(t)e^{i\phi(t)} \]

where

\[ a(t) = \sqrt{x^2(t) + H^2_x(t)} \]

and

\[
\phi(t) = \arg(H_x(t), x(t))
\]

\[
= \begin{cases} 
2 \arctan \frac{H_x}{\sqrt{H^2_x + x^2(t)}} & x > 0 \text{ or } y \neq 0 \\
\pi & x < 0 \text{ and } y = 0 \\
\text{Undefined} & x = 0 \text{ and } y = 0
\end{cases}
\]

The notion of instantaneous frequency and instantaneous slope of a signal's envelope, is then introduced as

\[ \omega(t) = \frac{d}{dt}\phi(t), \ \rho(t) = \frac{d}{dt}a(t) \]
Table 7.1 summarizes the transformations between elements of analytical and instantaneous representations of a analytic signal. These transformation were used to derive the augmented system (2.28).

\begin{align*}
\phi &= \arg(H_x, x) \\
a &= \sqrt{x^2 + H_x^2} \\
\omega &= \left(\frac{x}{H_x^2 + x^2} H_x - \frac{H_x}{H_x^2 + x^2}\right) \dot{x} \\
\rho &= \left(\frac{x^3 + xH_x^2}{(H_x^2 + x^2)^2} \right) H_x + \frac{x}{(x^2 + H_x^2)^2} \dot{x} \\
\dot{\omega} &= \left(\frac{x^3 + xH_x^2}{(H_x^2 + x^2)^2} \right) H_x - \frac{x^2 H_x + H_x^4}{(H_x^2 + x^2)^2} \ddot{x} \\
\dot{\rho} &= \frac{x^2 H_x + H_x^4}{(H_x^2 + x^2)^2} H_x + \frac{x^3 + xH_x^2}{(H_x^2 + x^2)^2} \dot{x} + \frac{(xH_x^2 - \dot{H}_x)^2}{(H_x^2 + x^2)^2} \\
x &= a \cos \phi \\
H_x &= a \sin \phi \\
\dot{x} &= \rho \cos \phi - a \omega \sin \phi \\
\dot{H}_x &= \rho \sin \phi + a \omega \cos \phi \\
\ddot{x} &= (\dot{\rho} - a \omega^2) \cos \phi - (a \omega^2 + 2 \rho \omega) \sin \phi \\
\ddot{H}_x &= (\dot{\rho} - a \omega^2) \sin \phi + (a \omega^2 + 2 \rho \omega) \cos \phi
\end{align*}

Table 7.1: Transformations to instantaneous coordinates

7.3 Appendix 3: Proof of Theorem 4.1

The general form provided in Eq. 4.9 implies that for each distinct Floquet exponent \( \mu_i \) there exists at least one eigensolution of the linear system in Eq. 4.3 of the form

\[ \tilde{z}_i(t) = e^{\mu_i t} q_i(t), \quad i \in \mathbb{N}, \quad i \leq 2n. \]

Note that, in the case of algebraic multiplicity of \( m_i > 1 \), the eigensolution \( \tilde{z}_{i,1} \) is of the above from.

Consequently, one can assume that each distinct Floquet exponent \( \mu_i \) of the linear system in Eq. 4.3, when it is rearranged in its second-order-form as
\[ \ddot{u} = J_f(\bar{x}) \dot{u}, \]  

(7.2)

is associated with an eigensolution of the form

\[ \tilde{u}_i = e^{\mu_i t} p_i(t), \quad i \in \mathbb{N}, \quad i \leq 2n, \]  

(7.3)

where \( \tilde{z}_i = \begin{bmatrix} \tilde{u}_i \\ \tilde{v}_i \end{bmatrix} \).

From the Eq. (7.2) and using the chain rule, we can obtain

\[ \delta \ddot{\tilde{u}}_i(t) = \delta \left[ J_f(\bar{x}) \right] \dot{\tilde{u}}_i(t) + J_f(\bar{x}) \delta \dot{\tilde{u}}_i(t). \]  

(7.4)

Also, it’s easy to see that

\[ \tilde{u}_i(\alpha + t) = e^{\mu_i t} \tilde{u}_i(\alpha), \quad \alpha = kT, \quad k \in \mathbb{Z}. \]  

(7.5)

In this way we can obtain

\[ \delta \tilde{u}_i(\alpha + t) = e^{\mu_i t} \delta \tilde{u}_i(\alpha) + t\delta \mu_i \tilde{u}_i(\alpha + t). \]  

(7.6)

and its second time derivative \( \delta \ddot{\tilde{u}}_i(\alpha + t) = \frac{d^2}{dt^2} [\tilde{u}_i(\alpha + t)] \) as

\[ \delta \ddot{\tilde{u}}_i(\alpha + t) = \mu_i^2 e^{\mu_i t} \delta \tilde{u}_i(\alpha) + t\delta \mu_i J_f(\bar{x}) \tilde{u}_i(\alpha + t) + 2\delta \mu_i \tilde{v}_i(\alpha + t). \]  

(7.7)

**Case I (Finding \( \delta \mu_i^{(1)} \))** - Respectively, from equations (7.4) and (7.7) we can calculate

\[ \delta \ddot{\tilde{u}}_i^{(1)}(T) = \delta \left[ J_f(\bar{x}) \right] \tilde{u}_i^{(1)}(T) + J_f(\bar{x}) \delta \tilde{u}_i^{(1)}(T) \]

and
\[
\delta \ddot{u}_i^{(1)}(T) = \left[ \mu_i^{(1)} \right]^2 e^{\mu_i^{(1)} T} \delta \dot{u}_i^{(1)}(0) + T \delta \mu_i^{(1)} \mathcal{J}_x(x) \dot{u}_i^{(1)}(T) + 2 \delta \mu_i^{(1)} \ddot{v}_i^{(1)}(T)
\]
which, since \( \ddot{u}_i^{(1)}(T) = e^{\mu_i^{(1)} T} \ddot{u}_i^{(1)}(0) \), \( \ddot{v}_i^{(1)}(T) = e^{\mu_i^{(1)} T} \ddot{v}_i^{(1)}(0) \) and (from Eq. (7.6)) \( \delta \ddot{u}_i^{(1)}(T) = e^{\mu_i^{(1)} T} \delta \ddot{u}_i^{(1)}(0) \), can be further simplified as

\[
\delta \ddot{u}_i^{(1)}(T) = \left\{ e^{\mu_i^{(1)} T} \left[ \left[ \partial_{x} f_x(x) \right] \dot{u}_i(0) + e^{\mu_i^{(1)} T} \mathcal{J}_x(x) \right] \delta \dot{u}_i(0) + T \delta \mu_i \ddot{v}_i(0) \right\}^{(1)}
\]

and

\[
\delta \ddot{v}_i^{(1)}(T) = \left\{ e^{\mu_i^{(1)} T} \left[ \mu_i^2 \delta \dot{u}_i(0) + T \delta \mu_i \mathcal{J}_x(x) \dot{u}_i(0) + 2 \delta \mu_i \ddot{v}_i(0) \right] \right\}^{(1)}.
\]

In this regard, from the above two equations for \( \delta \ddot{u}_i^{(1)}(T) \) and after substituting \( \zeta^{(1)} = \zeta + \delta \zeta^{(1)} \) and omitting higher order terms, one can calculate \( \delta \mu_i^{(1)} \) as

\[
\delta \mu_i^{(1)} = \frac{1}{2} \frac{\left\{ \left[ \partial_{x} f_x(x(T)) \right] \dot{u}_i \right\} \delta T + \left[ \mathcal{J}_x(x(T)) - \mu_i^2 I \right] \delta \dot{u}_i}{\| \dot{v}_i \|^2}.
\]

**Case II (Finding \( \delta \mu_i^{(2)} \))** - Similarly, in order to find \( \delta \mu_i^{(2)} \) we use both equations (7.4) and (7.7) with the difference that \( \delta T^{(2)} = 0 \) \( \delta \dot{u}_i(0) = 0 \) and \( \delta \ddot{u}_i^{(2)}(T) = h \). This \( \delta \mu_i^{(2)} \) can be readily obtained as

\[
\delta \mu_i^{(2)} = \frac{\left\{ \left[ \mathcal{J}_x(x) \dot{u}_i + 2 \ddot{v}_i \right] \right\} \left[ \mathcal{J}_x(x) h \right]}{\left\| \mathcal{J}_x(x) \dot{u}_i + 2 \ddot{v}_i \right\|^2}. \]

### 7.4 Appendix 4: Damped, Forced Systems

The proposed algorithm can be equally applied to equations of motion of forced, damped systems, such as a rotating machine or a wind turbine. In this regard, consider the damped, nonlinear system

\[
\dot{z} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} y \\ f(x, y) \end{bmatrix}
\]

(7.10)
that is exited by the forcing term \( g(t) \). The resultant forced system can be expressed by

\[
\dot{w} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{t} \end{bmatrix} = \begin{bmatrix} y \\ f(x, y) + g(t) \\ 1 \end{bmatrix} = G(w, t)
\] (7.11)

which for some periodic forcing terms such as \( \bar{g}(t) \) with the period \( T \), can accept a periodic solution \( \bar{w} \) with the same period. One can linearize the above system into

\[
\dot{\tilde{w}} = J_G(\bar{w}) \tilde{w}, \quad \tilde{w} = w - \bar{w}
\] (7.12)

where

\[
J_G(\bar{w}) = \begin{bmatrix} 0_{n \times n} & I_{n \times n} & 0_{n \times 1} \\ J^f_x(\bar{w}) & J^f_y(\bar{w}) & J^g(t) \\ 0 & 0 & 0 \end{bmatrix}
\] (7.13)

and consequently recast it as a forced time-periodic function

\[
\dot{\tilde{z}} = \begin{bmatrix} 0_{n \times n} & I_{n \times n} \\ J^f_x(\bar{w}) & J^f_y(\bar{w}) \end{bmatrix} \tilde{z} + \begin{bmatrix} 0_{n \times 1} \\ \dot{\bar{g}}(\bar{t}) \end{bmatrix}
\] (7.14)

where the matrix \( A \) is periodic with the same period \( T \). For the purpose of stability and bifurcation analysis, only the homogenous part is of interest. Therefore, we consider the homogenous system

\[
\dot{\tilde{z}} = \begin{bmatrix} 0_{n \times n} & I_{n \times n} \\ J^f_x(\bar{w}) & J^f_y(\bar{w}) \end{bmatrix} \tilde{z}
\] (7.15)

and reiterate Theorem 4.1.

### 7.5 Appendix 5: AbsTol and RelTol of Integration

In order to show that choosing \( \text{RelTol} = \text{AbsTol} \) will guarantee that \( h = \max\{\text{RelTol} \cdot |x|, \text{AbsTol}\} = \text{AbsTol} \) we must show that in each vector \( z_i(T) \) there is at least one component that has a smaller
absolute value than one, i.e.

\[ \forall i \in S, \exists j \in S, |z_{ij}(T)| \leq 1, \ S = \{1, \ldots, 2n\}. \tag{7.16} \]

Note that while evaluating the integration error, we're only interested in the error \( z_i(T) \) because it represents the accumulated integration error over one period and also because, if \( Z(0) \) is chosen to be the identity matrix, then the Monodromy matrix will be \( B = Z(T) \). To prove the statement in Eq. (7.16), we'll prove a stronger statement. In other words, we'll show that the largest possible absolute value for the smallest component \( \bar{z}_{ij}(T) \) is smaller than one, i.e.

\[ \forall i \in S, \max_i \{|z_{ij}(T)|\} \leq 1, \ S = \{1, \ldots, 2n\}. \tag{7.17} \]

We’ll prove the above statement for periodic solutions where all Floquet multipliers are very close to one (very small exponents). Then, for periodic solutions, where at least one pair but not all Floquet multipliers are very close to one, we’ll provide an upper limit for the Floquet multipliers so that the above statement remains valid.

First we’ll show that, the Monodromy matrix, for second order oscillatory systems with very small Floquet exponents (very close to one Floquet multipliers) becomes a rotation matrix. To explain this, let’s start with Eq. (4.7) where \( Z(0) \) is chosen to be the identity matrix, i.e.

\[ B = Z(T). \]

Since, for a second order oscillatory system, Floquet exponents are in the form of symmetric pairs of complex conjugates, then

\[ \det (B) = \det [Z(T)] = \prod_{i=1}^{2n} \rho_i = \left( \prod_{i=1}^{n} e^{-\phi_i - i\psi_i} \right) \left( \prod_{i=1}^{n} e^{\phi_i + i\psi_i} \right) = 1, \]
and hence, $Z(T)^{-1} = Z(T)$. Furthermore, if all the Floquet multipliers are very close to one, i.e. $ho_i \approx 1$, we’ll also have $Z(T)^T \approx Z(T)$ and therefore

$$Z(T)^T \approx Z(T)^{-1},$$

or in other words, the Monodromy matrix becomes a rotation matrix.

Next, we calculate the Euclidean (Frobenius) norm of the monodromy matrix $Z(T)$

$$\|Z(T)\|_F = \sqrt{\sum_{i=1}^{2n} \sum_{j=1}^{2n} \left|z_{ij}(T)\right|^2}$$

and use the equality

$$\sqrt{\sum_{i=1}^{2n} \sum_{j=1}^{2n} \left|z_{ij}(T)\right|^2} = \{\text{Tr} \left[Z(T)^T Z(T)\right]\}^{\frac{1}{2}}$$

to obtain

$$\sqrt{\sum_{i=1}^{2n} \sum_{j=1}^{2n} \left|z_{ij}(T)\right|^2} = \{\text{Tr} \left[Z(T)^T Z(T)\right]\}^{\frac{1}{2}} \approx \{\text{Tr}(I_{2n})\}^{\frac{1}{2}} = \sqrt{2n}.$$ 

To this end, since the sum of squared value of all components is constant, the largest possible value for the smallest component is achieved if all the components $z_{ij}(T)$ in the vector $z_{i}(T)$ approach the same value, we can write

$$2n \sum_{i=1}^{n} \left\{\left|z_{ij}(T)\right|^2\right\} = 4n^2 \left|z_{ij}(T)\right|^2 \approx 2n$$

or equivalently

$$\left|z_{ij}(T)\right| \approx \frac{1}{\sqrt{2n}} \leq 1, \text{ for } n \geq 1.$$ 

In case that at least one but not all of the Floquet exponents are very small, one can use
\[
\sqrt{\sum_{i=1}^{2n} \sum_{j=1}^{2n} (z_{ij}(T))^2} = \{ \text{Tr} \left[ Z(T)^T Z(T) \right] \}^{\frac{1}{2}} = \{ \text{Tr}(B^T B) \}^{\frac{1}{2}} = \left\{ \sum_{i=1}^{2n} \rho_i^2 \right\}^{\frac{1}{2}}.
\]

to find that

\[
|z_{ij}(T)| \approx \left\{ \sum_{i=1}^{2n} \rho_i^2 \right\}^{\frac{1}{2}} = \alpha_B
\]

to adjust

\[
\text{AbsTol} \times \alpha_B = \text{RelTol}.
\]

However, in this study, such an adjustment was not required as \(\alpha_B \leq 1\) for all periodic solutions with at least one pair of very small Floquet multipliers. In other words, for such periodic solutions we have

\[
\alpha_B = \left\{ \sum_{i=1}^{2n} \rho_i^2 \right\}^{\frac{1}{2}} \approx \frac{1}{4} \left( 1 + 1 + \bar{\rho} + \frac{1}{\bar{\rho}} \right)^{\frac{1}{2}} = \frac{1}{4} \sqrt{2 + (\bar{\rho} + \frac{1}{\bar{\rho}})}.
\]

Note that, \(\rho_1 = \rho_2 = 1\) and since \(\det(B) = 1\) the other two Floquet multiplier must satisfy \(\rho_3 \rho_4 = 1\) or \(\bar{\rho} = \rho_3 = \frac{1}{\rho_4}\). This means for all Floquet multipliers \(\bar{\rho} \leq 13.93\), or equivalently periodic solutions with (at least on pair of multipliers close to one and) the largest multiplier \(\bar{\rho} \leq 13.93\), no adjustment is required. In this study, for example, the largest \(\bar{\rho}\) in the branch of periodic orbits shown in Fig.4.4 is \(\bar{\rho} = 5.73\).

### 7.6 Appendix 6: Proof of Theorem 4.1

**Part A-** Let’s define the set

\[
E = \{ (t, x) | t \in (a,b), x \in \mathbb{R}^n \}.
\]  

(7.18)
and a set $E$ as a subset of the set $E$ satisfying conditions

(I) $E$ is an open set, 

(II) $\forall t_0 \in (a, b)$ the set $\{x|(t_0, x) \in E\}$ is nonempty and connected.

Next, let $x_{ij} \in \mathbb{R}$, $i = 1, \ldots, m$, $j = 1, \ldots, n$ be arbitrary real numbers. Since $f$ is continuous on $E$ and $E \in \mathbb{E}$ is an open set, then there exist numbers $a_1, b_1$ with $a \leq a_1 < t < b_1 \leq b$ and functions $x_1(t), \ldots, x_m(t)$ satisfying

$$
x_i(t_0) = [x_{ij}]_{j=1}^n \quad \dot{x}_i(t_0) = [y_{ij}]_{j=1}^n \quad \ddot{x}_i = f(x_i, \dot{x}_i)$$

(7.19)

where the points $\{x_1(t), \ldots, x_m(t)\}$ remain in $E$ for $a_1 < t < b_1$. Therefore the function

$$x(t) = F(x_1(t), \ldots, x_m(t), \dot{x}_1(t), \ldots, \dot{x}_m(t)), a_1 < t < b_1,$$

(7.20)

since $F$ is a connecting function, must satisfy

$$\dot{x} = f(x, \dot{x}), a_1 < t < b_1.$$  

(7.21)

and there must exist

$$x(t_0) = [x_{(m+1)j}]_{j=1}^n \quad \dot{x}(t_0) = [y_{(m+1)j}]_{j=1}^n.$$  

(7.22)

Next, in Part B, we shall express the derivative of $x$, i.e. $\dot{x}$ and $\ddot{x}$, in the partial derivative of $F$. In part C, we shall substitute the expression obtained for $\dot{x}$ and $\ddot{x}$ in to the Eq. (7.39). Using the resultant formula and condition (2) we then prove in Part D that $F$ is linear.

**Part B**- Since $F$ is a connecting function, the function $x(t)$ satisfies (7.39), in other words
\[
\frac{d^2}{dt^2} [F(t, x_1(t), \ldots, x_m(t))] = f(x, \dot{x})
\]  
(7.23)

which for an arbitrary \(1 \leq k \leq m\) yields

\[
\sum_{i=1}^{m} \sum_{j=1}^{n} \left[ \dot{x}_j^T \{ J_{x_i} (\nabla x_j F_k) \}^T \dot{x}_i \right] + \sum_{i=1}^{m} (\nabla x_i F_k) f_i = f(x, \dot{x}).
\]  
(7.24)

Taking \(t = t_0\) into the Eq. (7.24), we get

\[
\sum_{i=1}^{m} \sum_{j=1}^{n} \left[ ([y_{ij}]_{j=1}^{N})^T \{ J_{x_i} (\nabla x_j F_k) \}^T [y_{ij}]_{j=1}^{N} \right] + \sum_{i=1}^{m} (\nabla x_i F_k) f_i = f(x, \dot{x}).
\]  
(7.25)

**Part C**- Equation (7.25) implies

\[
|P(x_{ij}, y_{ij})| \leq |Q(x_{ij}, y_{ij})| + |f_k([x_{(m+1)j}]_{j=1}^{N}, [Y_{(m+1)j}]_{j=1}^{N})|
\]  
(7.26)

Using the condition (2), we shall show that \(P = 0\).

Let \(\xi_{ij}, i = 1, \ldots, m + 1, j = 1, \ldots, N\) be fixed real numbers and let \(\zeta > 1\) be a variable so that \(y_{ij} = \zeta \xi_{ij}\). Therefore, Eq. (7.26) yields

\[
|P(x_{ij}, \xi_{ij})| \leq |Q(x_{ij}, \xi_{ij})| + |f_k([x_{(m+1)j}]_{j=1}^{N}, [Y_{(m+1)j}]_{j=1}^{N})|
\]  
(7.27)

In the view of (7.25), it is clear that

\[
P(x_{ij}, \xi_{ij}) = \zeta^2 P(x_{ij}, \xi_{ij})
\]  
(7.28)

Thus, we can rewrite (7.27) as
\[ \zeta^2 |P(x_{ij}, \xi_{ij})| \leq |Q(x_{ij}, \zeta \xi_{ij})| + |f_k([x_{(m+1)j}]_{j=1}^{N}, \zeta [\xi_{(m+1)j}]_{j=1}^{N})|. \quad (7.29) \]

Defining

\[ N_k(x_{ij}, \zeta) = \max \left\{ |f_k(x_{ij}, y_{ij})| : |y_{ij}| \leq \omega \zeta^2, i = 1, \ldots, m + 1, j = 1, \ldots, N, \omega > 1 \right\}, \quad k = 1, \ldots, n \]

(7.30)

where, \( \omega \zeta = \max(y_{ij}) \), we obtain from Eq. (7.29) the inequality

\[ \zeta^2 |P(x_{ij}, \xi_{ij})| \leq O \left\{ \sum_{k=1}^{n} N_k(x_{ij}, \zeta) \right\} + |f_k([x_{(m+1)j}]_{j=1}^{N}, \zeta [\xi_{(m+1)j}]_{j=1}^{N})|. \quad (7.31) \]

Since,

\[ N_k(x_{ij}, \zeta) \leq \max \left\{ |f_k(x_{ij}, y_{ij})| : |y_{ij}| \leq \omega^2 \zeta^2, i = 1, \ldots, n, j = 1, \ldots, N, \omega > 1 \right\} = M(x_{ij}, \omega \zeta) \quad (7.32) \]

then

\[ \zeta^2 |P(x_{ij}, \xi_{ij})| \leq O \left\{ \sum_{k=1}^{n} M_k(x_{ij}, \zeta \omega) \right\} + O \left\{ M_k(x_{(m+1)j}, \zeta \omega) \right\}. \quad (7.33) \]

Dividing both sides of (7.31) by \( \zeta^2 \) and letting \( \zeta \to \infty \) we obtain

\[ |P(x_{ij}, \xi_{ij})| \leq \zeta^{-2} O \left\{ \sum_{k=1}^{n} M_k(x_{ij}, \zeta) \right\} + \zeta^{-2} O \left\{ M_k(x_{(m+1)j}, \zeta \omega) \right\}. \quad (7.34) \]

then,

\[ |P(x_{ij}, \xi_{ij})| \leq O \left\{ 1 \right\} + O \left\{ 1 \right\}, \text{ as } \zeta \to \infty. \quad (7.35) \]

Hence, \( P(x_{ij}, \xi_{ij}) = 0 \). It follows that the polynomial \( P(x, \dot{x}) \) vanishes identically.
Part D- Since $P = 0$, the coefficients $J_{x_i} \left( \nabla_{x_j} F_k \right)$ in Eq. (7.25) vanishes. Therefore, since $J_{x_i} \left( \nabla_{x_j} F_k \right)$ includes all and only all the second order partial derivative of the connecting function $F$, we conclude that the function $F$ has vanishing derivatives of order two. It follows that the function $F$ is a linear function of variables $x_i, i = 1, ..., m$.

7.7 Appendix 7: Proof of Theorem 4.2

Part A- Let’s define the set

$$E = \{(t, x)|t \in (a, b), x \in \mathbb{R}^n\}.$$  \hspace{1cm} (7.36)

and a set $E$ as a subset of the set $E$ satisfying conditions

(I) $E$ is an open set,

(II) $\forall t_0 \in (a, b)$ the set $\{x|(t_0, x) \in E\}$ is nonempty and connected.

Next, let $x_{ij} \in \mathbb{R}, i = 1, ..., m, j = 1, ..., n$ be arbitrary real numbers. Since $f$ is continuous on $E$ and $E \in \mathbb{E}$ is an open set, then there exist numbers $a_1, b_1$ with $a \leq a_1 < t_0 < b_1 \leq b$ and functions $x_1(t), ..., x_m(t)$ satisfying

$$x_i(t_0) = [x_{ij}]_{j=1}^n$$

$$\dot{x}_i(t_0) = [y_{ij}]_{j=1}^n$$

$$\ddot{x}_i = f(x_i, \dot{x}_i)$$  \hspace{1cm} (7.37)

where the points $\{x_1(t), ..., x_m(t)\}$ remain in $E$ for $a_1 < t < b_1$. Therefore the function

$$\phi(t) = \Phi(x_1(t), ..., x_m(t), \dot{x}_1(t), ..., \dot{x}_m(t)), a_1 < t < b_1,$$  \hspace{1cm} (7.38)

since $\Phi$ is a connecting function, must satisfy
\[ \ddot{\phi} = f(\phi, \dot{\phi}), \ a_1 < t < b_1. \]  

(7.39)

and there must exist \([x_{(m+1)j}]_{j=1}^n\) and \([y_{(m+1)j}]_{j=1}^n\) such that

\[ \phi(t_0) = [x_{(m+1)j}]_{j=1}^n \]

\[ \dot{\phi}(t_0) = [y_{(m+1)j}]_{j=1}^n. \]  

(7.40)

Next, in Part B, we shall express the derivative of \(\phi\), i.e. \(\dot{\phi}\) and \(\ddot{\phi}\), in terms of the partial derivative of \(\Phi\). In part C, we shall substitute the expression obtained for \(\dot{\phi}\) and \(\ddot{\phi}\) in to the Eq. (7.39). Using the resultant formula and condition (7.7) we then prove in Part D that \(\Phi\) satisfies (5.18).

**Part B-** Since \(\Phi\) is a connecting function, the function \(\phi(t)\) satisfies (7.39), in other words

\[ \frac{d^2}{dt^2} [\Phi(x_1(t), ..., x_m(t), \dot{x}_1(t), ..., \dot{x}_m(t))] = f(\phi, \dot{\phi}) \]  

(7.41)

Therefore for an arbitrary \(1 \leq k \leq n\)

\[ \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ x_j^T \{J_{x_j} (\nabla_x \Phi_k)\}^T \dot{x}_i + f_j^T \{J_{x_j} (\nabla_x \Phi_k)\}^T \dot{x}_i \right] + \sum_{i=1}^{n} (\nabla_x \Phi_k) f_i \]

\[ + \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ \dot{x}_j^T \{J_{x_j} (\nabla_x \Phi_k)\}^T f_i + f_j^T \{J_{x_j} (\nabla_x \Phi_k)\}^T f_i \right] + \sum_{i=1}^{n} (\nabla_x \Phi_k) \left[ \sum_{j=1}^{n} (J_{x_i} f_j) \dot{x}_j + \sum_{j=1}^{n} (J_{x_i} f_j) f_j \right] = f_k(\phi, \dot{\phi}). \]  

(7.42)

Taking \(t = t_0\) into the Eq. (7.42), we get

\[ \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ [y_{ij}]_{j=1}^N \right]^T \{J_{x_j} (\nabla_x \Phi_k)\}^T [y_{ij}]_{j=1}^N + f_j^T \{J_{x_j} (\nabla_x \Phi_k)\}^T [y_{ij}]_{j=1}^N \right] + \sum_{i=1}^{n} (\nabla_x \Phi_k) f_i \]

\[ + \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ [y_{ij}]_{j=1}^N \right]^T f_j^T \{J_{x_j} (\nabla_x \Phi_k)\}^T f_i + f_j^T \{J_{x_j} (\nabla_x \Phi_k)\}^T f_i \right] + \sum_{i=1}^{n} (\nabla_x \Phi_k) \left[ \sum_{j=1}^{n} (J_{x_i} f_j) [y_{ij}]_{j=1}^N + \sum_{j=1}^{n} (J_{x_i} f_j) f_j \right] = f_k(\phi, \dot{\phi}). \]  

(7.43)

which can be recast as

\[ P(x_{ij}, y_{ij}) + Q(x_{ij}, y_{ij}) + R(x_{ij}, y_{ij}) + S(x_{ij}, y_{ij}) = f_k(\phi, \dot{\phi}) \]  

(7.44)

in which
\[
\begin{align*}
P(x_{ij}, y_{ij}) &= \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ (y_{ji})_{i=1}^{N} \right]^T \left[ \left( Jx_j \nabla a_i \Phi_k \right) \right]^T [y_{ij}]_{j=1}^{N} \\
Q(x_{ij}, y_{ij}) &= \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ f_j^T \left( Jx_j \nabla a_i \Phi_k \right) \right]^T [y_{ij}]_{j=1}^{N} + \sum_{i=1}^{n} (\nabla a_i \Phi_k)^T f_i \\
R(x_{ij}, y_{ij}) &= \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ f_j^T \left( Jx_j \nabla a_i \Phi_k \right) \right]^T [y_{ij}]_{j=1}^{N} + \sum_{i=1}^{n} (\nabla a_i \Phi_k)^T \left[ \sum_{j=1}^{n} (Jx_j f)_j \right].
\end{align*}
\]

**Part C - Equation (7.57) implies**

\[
|P(x_{ij}, y_{ij})| \leq |Q(x_{ij}, y_{ij})| + |R(x_{ij}, y_{ij})| + |f_k(\Phi, \dot{\Phi})| \tag{7.46}
\]

Using (2) we shall show that \( P = 0 \).

Let \( \xi_{ij}, i = 1, \ldots, m+1, j = 1, \ldots, n \) be fixed real numbers and let \( \zeta > 1 \) be a variable so that \( y_{ij} = \zeta \xi_{ij} \).

Therefore, Eq. (7.59) yields

\[
|P(x_{ij}, \zeta \xi_{ij})| \leq |Q(x_{ij}, \zeta \xi_{ij})| + |R(x_{ij}, \zeta \xi_{ij})| + |f_k(x_{(m+1)j}, \zeta \xi_{(m+1)j})| \tag{7.47}
\]

Moreover, defining \( p \) as

\[
p = \max_s \left\{ s \left| \lim_{\zeta \to \infty} \zeta^{-s} L_i(\Phi_i, x, \zeta) = 0 \right| \right\} \tag{7.48}
\]

where, for any function \( v(x, \dot{x}) \)

\[
L_i(v, q, \zeta) = \min_{s} \{ |v(x, \dot{x})| : |\dot{x}_i| \leq \zeta, i = 1, \ldots, n \}, \tag{7.49}
\]

in the view of (7.58) and (7.48), one can conclude that

\[
P(x_{ij}, \zeta \xi_{ij}) \geq \zeta^2 \times \zeta^p P(x_{ij}, \xi_{ij}). \tag{7.50}
\]

Therefore,
\[ \zeta^{p+2} |P(x_{ij}, \xi_{ij})| \leq |Q(x_{ij}, \zeta \xi_{ij})| + |R(x_{ij}, \zeta \xi_{ij})| + |f_k(x_{(m+1)j}, \zeta \xi_{(m+1)j})| \]  
\hfill (7.51)

Next we define \( q \) and \( r \) as

\[
q = \max_s \left\{ s \mid \lim_{\zeta \to \infty} \zeta^{-s} M_i(\Phi_i, \mathbf{x}, \zeta) = 0 \right\}
\]
\[
r = \max_s \left\{ s \mid \lim_{\zeta \to \infty} \zeta^{-s} M_i(f_i, \mathbf{x}, \zeta) = 0 \right\}
\hfill (7.52)

where, for any function \( v(x, \dot{x}) \)

\[
M_i(v, q, \zeta) = \max\{|v(x, \dot{x})| : |\dot{x}_i| \leq \zeta, i = 1, ..., n\}.
\hfill (7.53)

Comparing equations (7.48) and (7.52) (also equations (7.49) and (7.53)), it is clear that \( p = q \) and according to the condition (2) \( r = 1, 2 \). Hence, Eq. (7.51) can be simplified to

\[
\zeta^{p+2} |P(x_{ij}, \xi_{ij})| \leq \zeta^{q} |Q(x_{ij}, \xi_{ij})| + \zeta^{p+2r-2} |R(x_{ij}, \zeta \xi_{ij})| + \zeta^r |f_k(x_{(m+1)j}, \zeta \xi_{(m+1)j})| \]  
\hfill (7.54)

Dividing both sides of (7.31) by \( \zeta^{p+2} \) and letting \( \zeta \to \infty \) we obtain

\[
|P(x_{ij}, \xi_{ij})| \leq \zeta^{q-p-2} |Q(x_{ij}, \xi_{ij})| + \zeta^{2r-4} |R(x_{ij}, \xi_{ij})| + \zeta^{r-p-2} |f_k(x_{(m+1)j}, \xi_{(m+1)j})| \]
\hfill (7.55)

\[
|P(x_{ij}, \xi_{ij})| \leq (\zeta^{-2} + \zeta^{2r-4} + \zeta^{r-p-2})O\{1\}, \text{ as } \zeta \to \infty.
\hfill (7.56)

then, if \( 2r - 4 \leq 0 \) or \( r = 1, 2 \) then \( P(x_{ij}, \xi_{ij}) = 0 \).

**Part D** - Since \( P = 0 \), the coefficients \( J_{x_i}(\nabla_{x_j} \Phi_k) \) in Eq. (7.57) vanishes. Therefore, since \( J_{x_i}(\nabla_{x_j} \Phi_k) \) includes all the second order partial derivative of the connecting function \( \Phi \), we conclude that the function \( \Phi \) has vanishing derivatives of order two with respect to the vectors \( x_i, i = 1, ..., m \). It follows that the function \( F \) is a linear function in the directions \( x_i, i = 1, ..., m \).
Part E- Let’s rearrange Eq. (7.42) into

\[ \bar{P}(x_{ij}, y_{ij}) + \bar{Q}(x_{ij}, y_{ij}) + \bar{R}(x_{ij}, y_{ij}) + \bar{S}(x_{ij}, y_{ij}) = f_k(\phi, \dot{\phi}) \] (7.57)

wherein

\[ \bar{P}(x_{ij}, y_{ij}) = R(x_{ij}, y_{ij}) \]
\[ \bar{Q}(x_{ij}, y_{ij}) = Q(x_{ij}, y_{ij}) \] (7.58)
\[ \bar{R}(x_{ij}, y_{ij}) = P(x_{ij}, y_{ij}). \]

Equation (7.57) implies that

\[ |\bar{P}(x_{ij}, y_{ij})| \leq |\bar{Q}(x_{ij}, y_{ij})| + |\bar{R}(x_{ij}, y_{ij})| + |f_k(\phi, \dot{\phi})|. \] (7.59)

Assuming that \( \xi_{ij}, i = 1, ..., m + 1, j = 1, ..., n \) be fixed real numbers and let \( \zeta > 1 \) be a variable so that, this time, \( x_{ij} = \zeta \xi_{ij} \). Therefore, Eq. (7.59) yields

\[ |\bar{P}(\zeta \xi_{ij}, y_{ij})| \leq |\bar{Q}(\zeta \xi_{ij}, y_{ij})| + |\bar{R}(\zeta \xi_{ij}, y_{ij})| + |f_k(\zeta \xi_{(m+1)j}, y_{(m+1)j})|. \] (7.60)

Moreover, let \( \bar{p}, \bar{q} \) and \( \bar{r} \) be

\[ \bar{p} = \max_s \left\{ s \left| \lim_{\zeta \to \infty} \zeta^{-s} L_i(\Phi_i, \zeta, \dot{x}) = 0 \right. \right\} \]
\[ \bar{r} = \max_s \left\{ s \left| \lim_{\zeta \to \infty} \zeta^{-s} M_i(\Phi_i, \zeta, \dot{x}) = 0 \right. \right\} \]
\[ \bar{q}_2 = \max_s \left\{ s \left| \lim_{\zeta \to \infty} \zeta^{-s} \bar{L}_i(f_i, \zeta, \dot{x}) = 0 \right. \right\} \]
\[ \bar{q}_1 = \max_s \left\{ s \left| \lim_{\zeta \to \infty} \zeta^{-s} \bar{M}_i(f_i, \zeta, \dot{x}) = 0 \right. \right\} \] (7.61)

where, for any function \( v(x, \dot{x}) \)

\[ \bar{L}_i(v, q, \zeta) = \min\{|v(x, \dot{x})| : |x_i| \leq \zeta, i = 1, ..., n\}. \]
\[ \bar{M}_i(v, q, \zeta) = \max\{|v(x, \dot{x})| : |x_i| \leq \zeta, i = 1, ..., n\}. \] (7.62)
One can readily show that \( \bar{p} = \bar{r} \) and \( \bar{q}_1 = \bar{q}_2 \). Therefore, similar to Part C, we can conclude that

\[
\zeta^{\bar{p} + 2\bar{q}_2} \left| \bar{P}(\zeta \xi_{ij}, y_{ij}) \right| \leq \zeta^{\bar{p} + \bar{q}_1 - 1} \left| \bar{Q}(\zeta \xi_{ij}, y_{ij}) \right| + \zeta^{\bar{p} - 2} \left| \bar{R}(\zeta \xi_{ij}, y_{ij}) \right| + \zeta^{\bar{q}_1} \left| f_k(\zeta \xi_{(m+1)j}, y_{(m+1)j}) \right| \tag{7.63}
\]

Dividing both sides of (7.31) by \( \zeta^{\bar{p} + 2\bar{q}_2} \) and letting \( \zeta \to \infty \) we obtain

\[
\left| \bar{P}(x_{ij}, \xi_{ij}) \right| \leq \zeta^{-\bar{q}_1 - 1} \left| Q(x_{ij}, \xi_{ij}) \right| + \zeta^{-2\bar{q}_2 - 2} \left| R(x_{ij}, \xi_{ij}) \right| + \zeta^{-\bar{q}_1 - \bar{p}} \left| f_k(x_{(m+1)j}, \xi_{(m+1)j}) \right| \tag{7.64}
\]

\[
\left| \bar{P}(x_{ij}, \xi_{ij}) \right| \leq (\zeta^{-\bar{q}_1 - 1} + \zeta^{-2\bar{q}_2 - 2} + \zeta^{-\bar{q}_1 - \bar{p}})O\{1\}, \text{ as } \zeta \to \infty. \tag{7.65}
\]

therefore, \( \bar{P}(x_{ij}, \xi_{ij}) = 0 \).

**Part F**- Since \( P = 0 \), the coefficients \( f_j^T \left\{ \mathbb{J}_{x_j} (\nabla x_k \Phi_k) \right\}^T + (\nabla x_k \Phi_k)^T \mathbb{J}_{x_j} f \right| = 0 \) in Eq. (7.77) vanishes. Therefore, since according to the condition (2) \( \mathbb{J}_{x_j} f = 0, s = 1 \), then we can further simplify the result as

\[
\mathbb{J}_{x_j} (\nabla x_k \Phi_k) = 0, \quad s = 1 \tag{7.66}
\]

\[
f_j^T \left\{ \mathbb{J}_{x_j} (\nabla x_k \Phi_k) \right\}^T + (\nabla x_k \Phi_k)^T \left( \mathbb{J}_{x_j} f \right) = 0, \quad s = 2.
\]

In the case of \( s = 1 \), i.e. conservative systems, since \( \mathbb{J}_{x_j} (\nabla x_k \Phi_k) \) includes all the second order partial derivative of the connecting function \( \Phi \) with respect to the velocity vectors, we conclude that the function \( \Phi \) has vanishing derivatives of order two. It follows that the function \( \Phi \) is also linear with respect to the velocity vectors, i.e. the directions \( \dot{x}_i, i = 1, ..., m \).

### 7.8 Appendix 8: Proof of Theorem 4.3

Suppose \( \{(x_k, \dot{x}_k) | k \in \mathbb{N}\} \) represents an arbitrary set of solutions of the system. Therefore, as stated in Theorem 5.2, if a connecting function \( \Phi \) as a function of these solutions exists, then it must be of
the form

$$
\Phi(x_1, \ldots, x_k, \dot{x}_1, \ldots, \dot{x}_k) = d + \sum_{i=1}^{m} A_i x_i + \sum_{i=1}^{n} B_i x_i + \sum_{i=1}^{n} \sum_{j=1}^{m} x_i^T C_{ijk} \dot{x}_j e_k.
$$

(7.67)

Moreover, by definition, the function $\Phi$ is a connecting function of the system $f$, if and only if, it is a solution of the system. In other words, $\Phi$ is a connecting function of $f$, if and only if

$$
\frac{d^2 \Phi}{dt^2} = f(\Phi).
$$

(7.68)

Equation (5.19) can be derived directly from the above necessary and sufficient condition.

### 7.9 Appendix 9: Proof of Theorem 4.4

Since $x_i$’s are periodic, they can be represented as

$$
x_i = d_i + \sum_{k \in \mathbb{N}} \left[ a_i \cos \left( \frac{2k\pi}{T_i} \right) + b_i \sin \left( \frac{2k\pi}{T_i} \right) \right]
$$

(7.69)

which consists of three groups of terms, i.e.

$$
\int_{0}^{p_i p_j T} \sin(mp_j \bar{\omega} t) \sin(np_i \bar{\omega} t) dt
$$

$$
\int_{0}^{p_i p_j T} \cos(mp_j \bar{\omega} t) \cos(np_i \bar{\omega} t) dt
$$

$$
\int_{0}^{p_i p_j T} \cos(mp_j \bar{\omega} t) \sin(np_i \bar{\omega} t) dt
$$

(7.70)

where $T = p_i p_j T$ and $\bar{\omega} = \frac{2\pi}{T}$.

The above integrations are always zero unless there exist at least one term with $p_i m = np_j$. However, since $\gcd(p_i, p_j) = 1$, then $p_i m = np_j$ implies that $p_i = m \leq M$ and $p_j = n \leq M$ which contradict the assumption that at most one integer $p_l \leq K$, $l \in \{1, \ldots, m\}$. 
Next, suppose that the vector period is not commensurable, i.e. there exist at least two periodic solutions \( x_i \) and \( x_j, j \neq i \) that \( \gcd(p_i, p_j) \geq 2 \). Therefore, for these two solution, one can construct \( \tilde{\tau} = T \begin{bmatrix} p_i \\ p_j \end{bmatrix} \). The vector \( \tilde{\tau} \) can be further simplified as \( \tau = T \times \gcd(p_i, p_j) \begin{bmatrix} \bar{p}_i \\ \bar{p}_j \end{bmatrix} \) where \( \gcd(\bar{p}_i, \bar{p}_j) = 1 \) and therefore \( \begin{bmatrix} \bar{p}_i \\ \bar{p}_j \end{bmatrix} \leq \frac{1}{2} \begin{bmatrix} p_i \\ p_j \end{bmatrix} \leq \frac{k+1}{2} \leq K \) winch contradicts the second condition.

### 7.10 Appendix 10: A Note on Shaw and Pierre’s (Non-)Linear Superposition

As explained in Chapter 2, Shaw and Pierre’s definition is formulated to find the loci of the nonlinear modes. For example, two invariant manifold of the system in Eq. (5.11) at the origin (its fixed point) can be obtained as (calculations can be found in [123])

\[
\Gamma_1(u_1, v_1) = \begin{bmatrix} u_1 + \frac{1}{12} (2u_1^2 + 3v_1^2)u_1 \\ v_1 + \frac{1}{4} v_1^2 \end{bmatrix} + O(u_1^3, v_1^3), \quad \Gamma_2(u_2, v_2) = \begin{bmatrix} u_2 \\ v_2 + \frac{1}{12} (4u_2^2 + v_2^2) v_2 \end{bmatrix} + O(u_2^3, v_2^3) \quad (7.71)
\]

where \( u_i = x_1, v_i = \dot{x}_1, i = 1, 2 \).

![Figure 7.1: Two dimensional invariant manifolds. Left: Displacement components of nonlinear modes. Right: Velocity component of nonlinear modes.](image)

Next, by rearranging theses invariant manifolds into the form of
Pierre and Shaw introduced the coordinate transformation

\[
M(x_1, y_1, x_2, y_2) = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

and suggested that it transforms the coordinates of the system into the coordinates of the invariant manifolds. In other words, it non-linearly projects each point in the state space of the system onto the surface of the mentioned invariant manifolds. Therefore the transformation, if used to transform the system of equation (5.11), would generate a set of differential equations, each governing the dynamics of solutions on one invariant manifolds. In this case, it transforms the system (5.11) to the uncoupled system

\[
\begin{bmatrix}
\ddot{u}_1 \\
\ddot{u}_2
\end{bmatrix} = \begin{bmatrix}
-1 & 0 \\
0 & -3
\end{bmatrix} \begin{bmatrix}
u_1 \\
u_2
\end{bmatrix} - \begin{bmatrix}
\frac{2}{3} & 0 \\
0 & \frac{4}{15}
\end{bmatrix} \begin{bmatrix}v_1^3 \\
v_2^3
\end{bmatrix} + \begin{bmatrix}
\frac{1}{14} & 0 \\
0 & \frac{3}{52}
\end{bmatrix} \begin{bmatrix}u_1 \dot{u}_1^2 \\
u_2 \dot{u}_2^2
\end{bmatrix}.
\]
Hence, using the inverse of the mentioned transformation, one should be able to non-linearly use solutions from the invariant manifold to reconstruct any solution of the system. In this regard, Shaw and Pierre suggested that the nonlinear transformation can serve as a nonlinear superposition rule (connecting function) [122, 123, 28].

However, rearranging Eq.(7.73) yields

\[
\begin{bmatrix}
  u_1 \\
  v_1 \\
  u_2 \\
  v_2
\end{bmatrix} = \begin{bmatrix}
  u_1 + \frac{1}{12}(2u_1^2 + 3v_1^2)u_1 \\
  v_1 \\
  u_2 + \frac{1}{52}(10u_2^2 + 3v_2^2)u_2 \\
  v_2
\end{bmatrix} + \begin{bmatrix}
  -u_2 + \frac{1}{52}(10u_2^2 + 3v_2^2)u_2 \\
  v_2 \\
  -v_2 + \frac{3}{52}(4u_2^2 + v_2^2)v_2
\end{bmatrix}
\]

\[z = \mathbf{\Gamma}_1(u_1, v_1) + \mathbf{\Gamma}_2(u_2, v_2)\]  

(7.75)

which presumes that every solution of the system can be reconstructed simply as a sum of two equally participating nonlinear modes of vibration, each from one of the invariant manifolds. This implies that the nonlinear transformation (7.73) leads to a linear supposition (not a nonlinear one) with unity coefficients.

To validate the soundness of this assumption one may find the inverse of the Eq.(7.73) for the initial condition \(z = [a, b, 0, 0]^T\). This initial condition can represent the sum of initial conditions of two nonlinear modes while passing through the equilibrium of the system. In this case, the inverse can be obtained as

\[
\begin{bmatrix}
  u_1 \\
  v_1 \\
  u_2 \\
  v_2
\end{bmatrix} = \begin{bmatrix}
  a - u_2 \\
  \pm(\frac{6}{5}u_2^2 - \frac{52}{5})^{\frac{1}{2}} \\
  u_2 \\
  \mp(\frac{6}{5}u_2^2 - \frac{52}{5})^{\frac{1}{2}}
\end{bmatrix}
\]

(7.76)

where \(u_2\) is the real solution of \(u_2^3 - \left[39 \frac{34}{39} a\right]u_2^2 + \left[195 \frac{1394}{697}a^2 - \frac{5772}{697}\right]u_2 = \frac{65}{1397}a^3 - \frac{5889}{697}a - \frac{195}{697}b\) subject to the condition \(|u_2| \geq \sqrt{\frac{34}{3}}\). Consequently, one can readily show that Shaw and Pierre’s prediction of their corresponding nonlinear modes are
\[
\Gamma_1(u_1, v_1) = \begin{cases} 
\frac{a - u_2}{a - u_2 + \frac{1}{12}(2(a - u_2)^2 + 3(\frac{6}{5}u_2^2 - \frac{52}{5}))}(a - u_2) \\
\pm\left(\frac{6}{5}u_2^2 - \frac{52}{5}\right)^{\frac{1}{2}} \\
\pm\left[\left(\frac{6}{5}u_2^2 - \frac{52}{5}\right)^{\frac{1}{2}} + \frac{1}{4}\left(\frac{6}{5}u_2^2 - \frac{52}{5}\right)^{\frac{3}{2}}\right]
\end{cases}
\]

\[
\Gamma_2(u_2, v_2) = \begin{cases} 
-\frac{u_2}{u_2 + \frac{1}{52}(10u_2^2 + 3(\frac{6}{5}u_2^2 - \frac{52}{5}))u_2} \\
\pm\left(\frac{6}{5}u_2^2 - \frac{52}{5}\right)^{\frac{1}{2}} \\
\pm\left(\frac{6}{5}u_2^2 - \frac{52}{5}\right)^{\frac{1}{2}} \left[-1 + \frac{3}{62}(4u_2^2 + (\frac{6}{5}u_2^2 - \frac{52}{5}))\right]
\end{cases}
\]

Next, the mentioned assumption have been examined for initial conditions in the form of \(z = [a, b, 0, 0]^T\). Figure (7.2) compares the prediction results from both Shaw and Pierre's proposed superposition and the proposed linear approximation of connecting functions (using orthogonal periodic solutions) for the same set of initial conditions. Results reveals that not only does the suggested superposition rule fail to provide an accurate prediction of the solutions of the system, but because of the aforementioned condition on \(u_2\) in Eq.(7.76) and stability of the system (7.74), it can’t provide any prediction for initial conditions in a strongly nonlinear regions.
Figure 7.2: Part I-Left: nonlinear modes used in ICM superpositions show 8% and 20.3% increase in frequencies with respect to the linear frequencies of the system. Part I-Right: Comparison between superpositions found by the ICM approach versus the one proposed by Pierre and Shaw. Part II-Left: 18.2% and 23.9% increase in frequencies puts the selected nonlinear modes in a fairly nonlinear region. Part II-Right: Three is only one valid solution for superposition for Eq.(7.76). Part III-Left: Shows 78.8% and 35.4% increase in frequencies. Part III-Right: Equation (7.74) is unstable for the only valid solution of the Eq.(7.76). Only the superposition rules (obtained by ICM approach) are show for which a Pierre and Shaw’s superpositions also exist.
Therefore, as illustrated in Fig. (7.2), one can readily observe that the original assumption that the nonlinear system accept a connecting function of the form $\Phi(q_1, q_2) = q_1 + q_2$, as predicted by Spijker, is simply not valid and consequently the nonlinear transformation in (7.73) does not represent a projection of initial conditions of the system onto its invariant manifolds.
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